# Mentorship Program for Summer 2025
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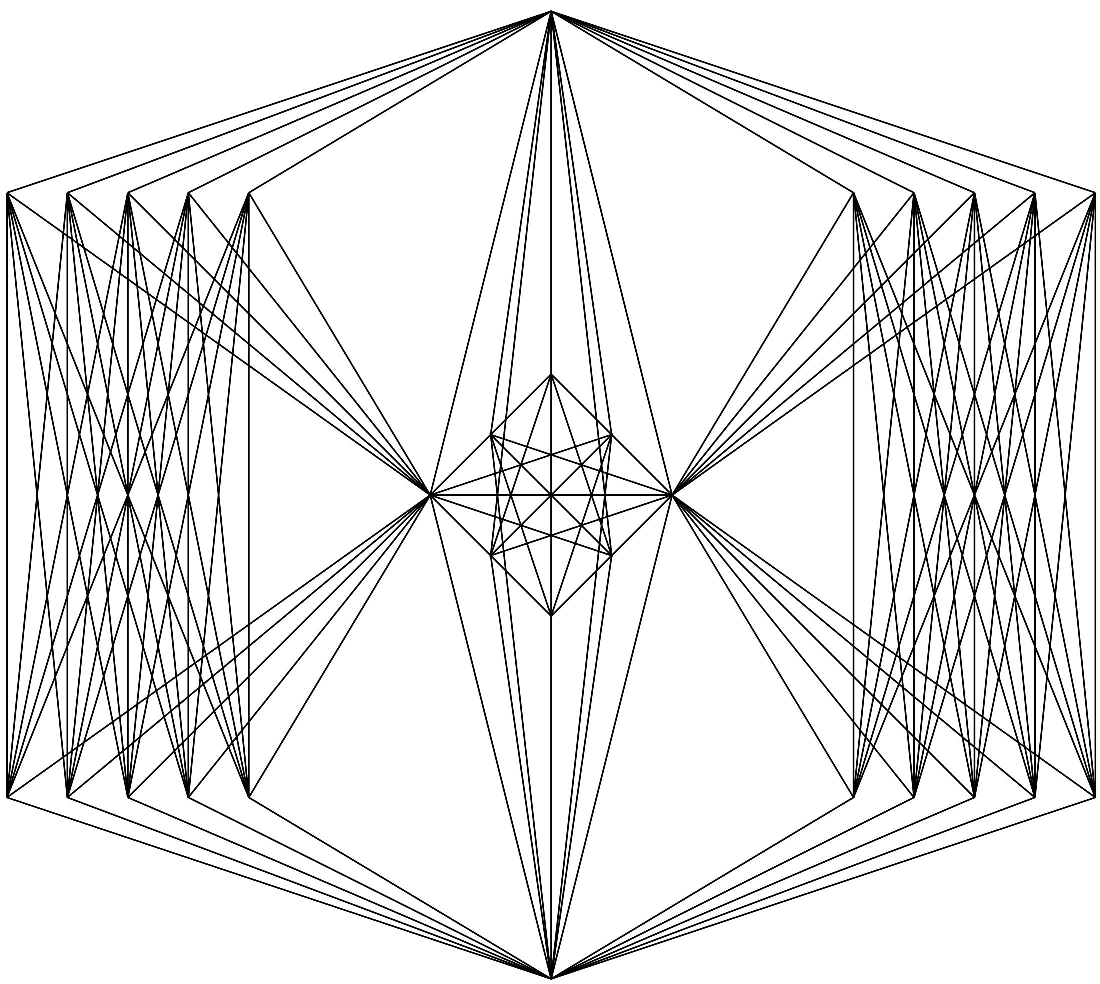


# 1 Introduction

The goal of this paper is to provide the necessary information required to provide a successful math jam through the Society of Undergraduate Mathematics - Calgary based on the work done by myself and Kristaps Balodis during the summer of 25.

During the begining of the summer Kristaps and I decided that I should try and learn some Algebraic Graph Theory. To this end we landed on reading "Spectral Graph Theory, Expanders, and Ramanujan Graph" by Christopher Williamson, 2014.

There is much to be discussed in that paper, most of which I did not have time to cover. However, for the purposes of this preparation document we will be focussing on what Williamson called the "Cheeger Constant" of a graph.

The first introduction to the ideal of the Cheeger Constant was in section 2 of the thesis, but in section 3, Williamson connects the idea of the Cheeger constant to many important graph theoretic values. One example includes the chromatic number of a graph. But the most imediate use of the cheeger constant is the lower and upper bound it provides to the second smallest eigenvalue of the normalized Laplacian matrix of a graph.

In this document we will be focusing on finding and proving the value of the cheeger constant of a path matrix.

# 2 Definitions

**Definition 1:** For our purposes a graph $G = (V,E)$ is a pair of vertices and simple (non-directional, non-loop, non-multi, two ended) edges. Each edge will be denoted by a set containing two elements.

**Definition 2:** A path graph of length $n \in \mathbb{Z}_{>0}$ is a graph $P_n = (V,E)$ where there exists a labeling of the graph $P_n$ such that $V = \{1,2,\ldots,n\}$ and $E = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\}\}$.

**Definition 3:** For a graph $G = (V,E)$ we define the degree of a vertex $v \in V$ to be

$$\deg_G(v) = |\{w \in V \mid \{v,w\} \in E\}|$$

If the context is clear, we remove the subscript $G$.

**Definition 4:** For a given path graph $P_n = (V,E)$ the end vertices of $P_n$ are the vertices in $V$ that have degree 1.

**Definition 5:** For a given graph $G$ we define the component set of $G$ by $\mathcal{C}(G) = \{\text{connected components of } G\}$.

**Remark 6:** We take the definition of a connected component as given.

**Definition 7:** For a graph $G = (V,E)$, we define the complement of $G$ to be $\overline{G} := (V,\overline{E})$ where

$$\overline{E} := \{\{v,w\} \mid v,w \in V, \{v,w\} \text{ is a simple edge, and } \{v,w\} \notin E\}$$

**Definition 8:** For a given graph $G = (V,E)$ and a given subset $X \subseteq V$ we define the induced subgraph of $X$ in $G$ to be $\mathcal{G}(X) := (X, E_X)$ where

$$E_X = \{\{v,w\} \in E \mid v,w \in X\}$$

**Definition 9:** For a given graph $G = (V,E)$ and a given subset $X \subseteq V$, we define the graph complement of the set $X$ by $\overline{X} := \overline{\mathcal{G}(X)}$, by an abuse of notation.

**Remark 10:** Note that $X$ and $\overline{X}$ can simultaneously be considered as sets and as induced subgraphs by abuse of notation.

**Definition 11:** For a graph $G = (V,E)$ we define the volume of a subset $X \subseteq V$ to be

$$\text{vol}_G(X) = \sum_{x \in X} \deg_G(x)$$

If the context is clear we may remove the subscript $G$.

**Definition 12:** For a graph $G = (V,E)$ we define the intermediate edges of subsets $X, Y \subseteq V$ to be

$$E(X,Y) := \{\{v,w\} \in E \mid v \in X, w \in Y\}$$

**Definition 13:** For a graph $G = (V,E)$ we define the Cheeger constant on a vertex subset $X \in \mathcal{P}^*(V)$ to be the value

$$h_G(X) := \frac{|E(X,\overline{X})|}{\min\{\text{vol}_G(X), \text{vol}_G(\overline{X})\}}$$

where $\mathcal{P}^*(V) = \mathcal{P}(V) \setminus \{\emptyset, V\}$, and $\mathcal{P}(V)$ is the power set of $V$.

**Definition 14:** For a graph $G = (V,E)$ we define the Cheeger constant of $G$ to be the value

$$h_G := \min_{X \in \mathcal{P}^*(V)} h_G(X)$$

## 3 Approach

As stated, the goal of this document is to get prepared to prove the value of the Cheeger constant of a path graph for an audience during a math jam. As the goal of doing a math jam is to practice teaching, I would like to focus on "How do you approach a problem?" This is a particular question that Dr. Ryan Hamilton suggested to me last year's winter semester. He suggested that SUMC should do regular events where upper year individuals show their thinking process when attacking a math problem. This way the lower year students can take away what we have learned over the years and develop their own methods of approaching problems, making them better mathematicians in the process.

So, let us go through my thinking process of how I approach this problem (and even problems in general).

### 3.1 Identify the problem

When ever I wish to approach a problem, the first thing that I do is to try to understand what the problem is trying to communicate. What that means depends on the person them self, but to me that means the following:

1. Write the statement down in mathematical terms.
2. Simplify the statement to its more core elements.
3. *(Optional)* Write/say the problem in normal words.

Writing the statement down in mathematical form allows me to be more concrete with how I approach the problem. It is too easy for me to get lost in the nuances of the problem. Particularly with this one, I had a tendency to jump to what my gut/intuition says. While that is an important element of problem solving, I find that my gut is my enemy when it comes to the first step of approaching a problem. The reason I say this is because my gut may have an incing about how to approach the problem, and that is great! However, how can I approach a problem which I do not understand? Writing the question down as a mathematical statement helps me ground myself, my intuition, and my mind.

Afterwards, I break apart the statement. I look at each and every component of the statement. If there is a word with a loaded definition, then I expand it, and write the statement down again. This will help me understand logically what my approach should be. This will either reinforce the gut feeling that I have been having or completely demolish it. Either way, this step is what actually helps me understand what the next steps should be from a logical point rather than a intuitive point.

And lastly, if I am still having a hard time making the question feel ok in my gut (aka in the case where I never had any intuition) I like to write down or say the statement in my own non-mathematical words. This way I get a sense of the question within myself, rather than just within my pen and paper.

So, if I try to write down the goal of this document in a mathematical way.

Looking at the question at hand, we know that we are looking for a particular value ... but looking for a value is not a mathematical statement. Currently our statement is “The Cheeger constant of the path graph of length $n$ is ...”. That is not yet a mathematical statement, because it is not yet a sentence. Hence our goal becomes clear! We need to identify an intuition for what the Cheeger constant of $P_n$ is, then try to prove it!

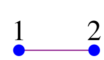
### 3.2 Examples

My method for building an intution is to either understand all important factors, or do examples. I like examples more. Especially when the problem is discrete!

**Example 15:** Let $n = 1$. Note that in this case we have a labeling $P_1 = (V = \{1\}, E = \emptyset)$. However, the cheeger constant of a graph looks at subset of the vertex set that is not all the vertices and not the empty set. Such a subset of $\{1\}$ does not exist. Hence we know that the problem does not make sense in this case.

**Example 16:** Let $n = 2$. Consider $P_2 = (V = \{1, 2\}, E = \{1, 2\})$.

$P_2$ :



Note that in this case $\mathscr{P}^*(V) = \{\{1\},\{2\}\}$. Now to build our self a table to understand what is happening we have the following:

| $X$ | $\overline{X}$ | $\text{vol}(X)$ | $\text{vol}(\overline{X})$ | $E(X,\overline{X})$ | $h_{P_2}(X)$ |
|---|---|---|---|---|---|
| $\{1\}$ | $\{2\}$ | 1 | 1 | 1 | 1 |
| $\{2\}$ | $\{1\}$ | 1 | 1 | 1 | 1 |

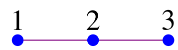
Hence we conclude that $h_{P_2} = 1$.

Note that this is telling us that there is a symmetry happening. In order to identify why, we have to go back to the definition to understand why such a symmetry is occurring.

Well, if we go back to the definition we see that the fraction in the definition of a Cheeger constant is completely symmetric with respect to complement sets of vertices! That is our first Claim!

**Claim:** $h_G(X) = h_G(\overline{X})$ for any graph $G = (V,E)$ and subset $X \in \mathscr{P}^*(V)$.

**Example 17:** Consider the graph

$P_3$ : 


Note that in this case $V = \{1,2,3\}$ and $E = \{\{1,2\},\{2,3\}\}$. Furthermore, note that

$$\mathscr{P}^*(V) = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{1,3\}\}$$

Now (following the intution of the claim above) we set an equivalence relation on $\mathscr{P}^*(V)$ such that for $X,Y \in \mathscr{P}^*(V)$, $X \sim Y$ if and only if $Y = \overline{X}$. It is easy to see that this is infact an equivalence relation. Note that now we have

$$\mathscr{P}^*(V)/\sim = \{\{1\},\{2\},\{3\}\}$$

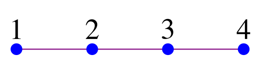
And this literally divided the $X$ we need to check by two! Let us again look at our table in this case:

| $X$ | $\overline{X}$ | $\text{vol}(X)$ | $\text{vol}(\overline{X})$ | $E(X,\overline{X})$ | $h_{P_3}(X)$ |
|---|---|---|---|---|---|
| $\{1\}$ | $\{2,3\}$ | 1 | 3 | 1 | 1 |
| $\{2\}$ | $\{1,3\}$ | 2 | 2 | 2 | 1 |
| $\{3\}$ | $\{1,2\}$ | 1 | 3 | 1 | 1 |

Hence, we again conclude that $h_{P_3} = 1$.

I am having some hard time understanding why there was a difference between the values and how to generalize. So lets do another example!

**Example 18:** Consider the graph

$P_4$ : 

In this case, using the same equivalence relation as before we have

$$\mathscr{P}^*(V)/\sim = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\}\}$$

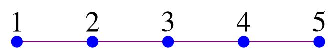
In this case our table looks like the following:

| $X$ | $\overline{X}$ | $\mathrm{vol}(X)$ | $\mathrm{vol}(\overline{X})$ | $E(X,\overline{X})$ | $h_{P_4}(X)$ |
|---|---|---|---|---|---|
| $\{1\}$ | $\{2,3,4\}$ | 1 | 5 | 1 | 1 |
| $\{2\}$ | $\{1,3,4\}$ | 2 | 4 | 2 | 1 |
| $\{3\}$ | $\{1,2,4\}$ | 2 | 4 | 2 | 1 |
| $\{4\}$ | $\{1,2,3\}$ | 1 | 5 | 1 | 1 |
| $\{1,2\}$ | $\{3,4\}$ | 3 | 3 | 1 | $1/3$ |
| $\{1,3\}$ | $\{2,4\}$ | 3 | 3 | 3 | 1 |
| $\{1,4\}$ | $\{2,3\}$ | 2 | 4 | 2 | 1 |

Hence, we conclude that $h_{P_4} = \frac{1}{3}$.

WHAT THE HECK? It changed up on us! Lets do another example!

**Example 19:** Consider the graph

$P_5$ : 


In this case, using the same equivalence relation as before we have

$$\mathscr{P}^*(V)/\sim = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}$$

In this case our table looks like the following:

| $X$ | $\overline{X}$ | $\mathrm{vol}(X)$ | $\mathrm{vol}(\overline{X})$ | $E(X,\overline{X})$ | $h_{P_5}(X)$ |
|---|---|---|---|---|---|
| $\{1\}$ | $\{2,3,4,5\}$ | 1 | 7 | 1 | 1 |
| $\{2\}$ | $\{1,3,4,5\}$ | 2 | 6 | 2 | 1 |
| $\{3\}$ | $\{1,2,4,5\}$ | 2 | 6 | 2 | 1 |
| $\{4\}$ | $\{1,2,3,5\}$ | 2 | 6 | 2 | 1 |
| $\{5\}$ | $\{1,2,3,4\}$ | 1 | 7 | 1 | 1 |
| $\{1,2\}$ | $\{3,4,5\}$ | 3 | 5 | 1 | $1/3$ |
| $\{1,3\}$ | $\{2,4,5\}$ | 3 | 5 | 3 | 1 |
| $\{1,4\}$ | $\{2,3,5\}$ | 3 | 5 | 3 | 1 |
| $\{1,5\}$ | $\{2,3,4\}$ | 2 | 6 | 2 | 1 |
| $\{2,3\}$ | $\{1,4,5\}$ | 4 | 4 | 2 | $1/2$ |
| $\{2,4\}$ | $\{1,3,5\}$ | 4 | 4 | 4 | 1 |
| $\{2,5\}$ | $\{1,3,4\}$ | 3 | 5 | 3 | 1 |
| $\{3,4\}$ | $\{1,2,5\}$ | 4 | 4 | 2 | $1/2$ |
| $\{3,5\}$ | $\{1,2,4\}$ | 3 | 5 | 3 | 1 |
| $\{4,5\}$ | $\{1,2,3\}$ | 3 | 5 | 1 | $1/3$ |

Hence we conclude that $h_{P_5} = \frac{1}{3}$.

Note that we can clearly see a connection between the following sets:

$$\begin{aligned}
\mathcal{X}_1 &= \{\{1\},\{5\}\} \\
\mathcal{X}_2 &= \{\{2\},\{3\},\{4\}\} \\
\mathcal{X}_3 &= \{\{1,2\},\{4,5\}\} \\
\mathcal{X}_4 &= \{\{1,3\},\{1,4\},\{2,5\},\{3,5\}\} \\
\mathcal{X}_5 &= \{\{1,5\}\} \\
\mathcal{X}_6 &= \{\{2,3\},\{3,4\}\} \\
\mathcal{X}_7 &= \{\{2,4\}\}
\end{aligned}$$

This separation is telling us a few things matter when we are talking about this problem:

1. Does your set $X$ include the end vertices?
   (a) If so, is it just one, or both?
2. How many non-end-points are in your set?
3. Is the $\mathcal{G}(X)$ a connected graph?
   (a) If not, then how many components?

It is important to note that $\mathcal{X}_3$ was the one collection that resulted in the least value of $h_{P_5}$.

Lastly, it is important to note that there is a balance between $\mathrm{vol}(X)$ and $\mathrm{vol}(\overline{X})$.

Now, I do not yet have enough information to be able to identify what the actual value of the Cheeger constant is because I don't have enough data yet. But I seriously do not wanna go into $P_6$ right now, because my set $^{\mathcal{P}^*(V)}/_{\sim}$ is just too big. So for now, let us focus on the five questions stated above and the other patters we have noticed so far. This way we can actually prove something and hopefully make our life easier in the future.

### 3.3 Noticed Patterns

In Section 3.2 we saw a few patterns. Now, we need to make that more rigorous. Which patterns are simply coincidence, and which patterns are truly patterns? Once we have identified the patterns, we must prove them.

In this section, we prove the patterns that I noticed in my approach. For now, suppose that $G = P_n = (V,E)$. Some of the patterns we discuss are not specific to $P_n$ and those will be addressed.

**Theorem 20:** $\forall\ X \in \mathcal{P}^*(V),\ h_G(X) = h_G(\overline{X})$.

Proof: Note that for any $A,B \in \mathcal{P}(V)$,

$$E(A,B) = E(B,A) \tag{1}$$

since $\{v,w\} = \{w,v\}$ for any $v \in A$, $w \in B$. Furthermore, note that for any $a,b \in \mathbb{Z}$,

$$\min(a,b) = \min(b,a) \tag{2}$$

by definition. Hence for $X \in \mathscr{P}^*(V)$ we have the following:

$$\begin{aligned} h_G(X) &= \frac{|E(X,\overline{X})|}{\min(\text{vol}(X),\text{vol}(\overline{X}))} \\ &= \frac{|E(\overline{X},X)|}{\min(\text{vol}(X),\text{vol}(\overline{X}))} && \text{(by (1))} \\ &= \frac{|E(\overline{X},X)|}{\min(\text{vol}(\overline{X}),\text{vol}(X))} && \text{(by (2))} \\ &= h_G(\overline{X}) \end{aligned}$$

Thus we have proven the desired statement. ■

This is the claim the first claim that we had in Section 3.2.

**Remark 21:** Theorem 20 applies to all graphs $G$ not just $P_n$. The statement is dependent on $G$ being a non-directed graph.

If you recall. We also stated that there was a balance between vol$X$ and vol$\overline{X}$. Notice that every time they add up to $2|E|$. The resoning for this is due to a very elementary graph theory concept!

**Lemma 22:** For a graph $G$, the sum of the degrees of every vertex is equal to twice the number of edges.

Proof: Let $G = (V,E)$ be a simple graph. Suppose that we count the degree of every vertex. In the process of doing so, we had to count each edge from both end-vertices of the edge. Hence, by summing the degree of every vertex we have essentially counted every edge twice. Hence we conclude that

$$\sum_{v\in V} \deg_G(v) = 2|E|$$

Thus we conclude that we have proven the desired statement. ■

With this we are able to concretely state and prove the balance between vol$(X)$ and vol$(\overline{X})$!

**Theorem 23:** $\forall\, X \in \mathscr{P}^*(V)$, $\text{vol}(X) + \text{vol}(\overline{X}) = 2|E| = \text{vol}(V)$.

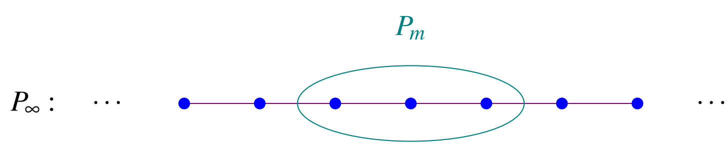
Proof: Let $X \in \mathscr{P}^*(V)$. Recall that $X \subseteq V$. It follows directly from the definition that $\overline{X} = V \setminus X$. Hence we know that $V = X \sqcup \overline{X}$. As such we have the following:

$$\begin{aligned} \text{vol}(X) + \text{vol}(\overline{X}) &= \sum_{v\in X} \deg_G(v) + \sum_{w\in \overline{X}} \deg_G(w) \\ &= \sum_{u\in X\sqcup\overline{X}} \deg_G(u) \\ &= \sum_{u\in V} \deg_G(u) \\ &= 2|E| && \text{(by Lemma 22)} \\ &= \text{vol}(V) && \text{(by definition)} \end{aligned}$$

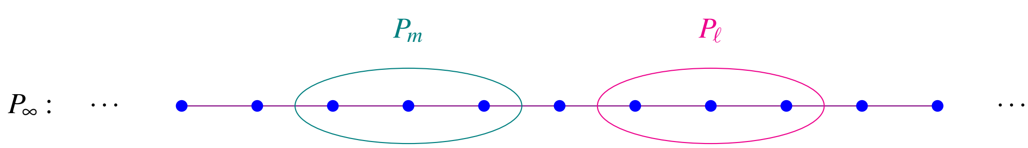
Thus we see that we have proven the desired statement. ■

Now let us get into more complected relations. To understand what is happening with $h_G$ we have to understand what happens with $h_G(X)$, and to do that we need to understand $E(X,\overline{X})$, $\min(\mathrm{vol}(X),\mathrm{vol}(\overline{X}))$, and how the two relate to each other.

Taking a look at $E(X,\overline{X})$ we have to visually think about what impacts its value. The first thing to notice is that every connected component of $X$ will be a graph of the form $P_m$ where $m \in \mathbb{Z}_{>0}$, $m < n$ (strict inequality since $X$ cannot be $V$). Now, the way I like to think about this is if I just focused on the $P_m$, and imagined that $P_m$ is contained within a countably infinite graph $P_\infty$:




Now note that we can't have two disconnected components of $X$, $P_m$ and $P_\ell$, touching. Otherwise they would be connected components. Hence we know that the edges coming out of the endpoints of $P_m$ are edges counted in $E(X,\overline{X})$.




Hence we can see that out of each connected component comes two end vertices. However, since we are not actually working with $P_\infty$ we have to take into account when we are looking at $P_n$. In this case it is possible for $X$ to contain 0, 1, or 2 end vertices of $P_n$. Each one of those cases changes the number of intermediate edges each connected component of $X$ has with $\overline{X}$. Taking all of this into account, we get the following theorem.

**Theorem 24:** For $X \in \mathscr{P}^*(V)$,

$$E(X,\overline{X}) = \begin{cases} 2|\mathcal{C}(X)|, & \text{if } X \text{ has no end vertices of } P_n \\ 2|\mathcal{C}(X)|-1, & \text{if } X \text{ has one end vertex of } P_n \\ 2|\mathcal{C}(X)|-2, & \text{if } X \text{ has both end vertices of } P_n \end{cases}$$

Proof: Let $X \in \mathscr{P}^*(V)$. Furthermore, let $a$ and $b$ be the two end vertices of $P_n$. We will prove the statement by looking at the distinct cases: $a,b \notin X$, $a \in X$ and $b \notin X$ or vice versa, and $a,b \in X$.

Case 1: Suppose $a \notin X$ and $b \notin X$. We have already discussed that $\mathcal{C}(P_n)$ is a set of graphs of the form $P_m$ where $m < n$. Since $a \notin X$ and $b \notin X$ we know that for any $P_m \in \mathcal{C}(X)$, $a \notin V(P_m)$ and $b \notin V(P_m)$. As such we know that

$$E(P_m,\overline{X}) = 2 \tag{3}$$

for all $P_m \in \mathcal{C}(X)$. Thus, we have the following:

$$E(X,\overline{X}) = \sum_{P_m \in \mathcal{C}(X)} E(P_m,\overline{X})$$

$$
\begin{aligned}
&= \sum_{P_m \in \mathcal{C}(X)} 2 && \text{(by (3))} \\
&= 2|\mathcal{C}(X)|
\end{aligned}
$$

Case 2: Suppose that either $a \in X$ and $b \notin X$, $a \notin X$ and $b \in X$. Without loss of generality suppose $a \in X$ and $b \notin X$. We know that there exists exactly one component of $X$ which contains $a$. Let $P_\ell$, for $\ell < n$, be that component. Note that we have

$$E(P_\ell, \overline{X}) = 1 \tag{4}$$

Furthermore, similar to Case 1 we know that for any other connected component $P_m \in \mathcal{C}(X)$ we have

$$E(P_m, \overline{X}) = 2 \tag{5}$$

With these pieces of information we have the following:

$$
\begin{aligned}
E(X, \overline{X}) &= \sum_{P_m \in \mathcal{C}(X)} E(P_m, \overline{X}) \\
&= E(P_\ell, \overline{X}) + \sum_{P_m \in \mathcal{C}(X) \setminus \{P_\ell\}} E(P_m, \overline{X}) \\
&= 1 + \sum_{P_m \in \mathcal{C}(X) \setminus \{P_\ell\}} E(P_m, \overline{X}) && \text{(by (4))} \\
&= 1 + \sum_{P_m \in \mathcal{C}(X) \setminus \{P_\ell\}} 2 && \text{(by (5))} \\
&= 1 + 2(|\mathcal{C}(X)| - 1) \\
&= 1 + 2|\mathcal{C}(X)| - 2 \\
&= 2|\mathcal{C}(X)| - 1
\end{aligned}
$$

Case 3: Suppose $a \in X$ and $b \in X$. Via the similar computations as Case 2 we see that $E(X, \overline{X}) = 2|\mathcal{C}(X)| - 2$.

We see that all the cases present are exhaustive and they identically match with the statement of the theorem. Hence we conclude that we have proven the theorem.

■

Now that we understand the behavior of $E(X, \overline{X})$ we can focus on $\text{vol}(X)$. Then hopefully we can understand the behavior of $h_G(X)$.

**Theorem 25:** For all $X \in \mathcal{P}^*(V)$ we have

$$
\text{vol}(X) = \begin{cases} 2|X| & \text{if } X \text{ has no end vertices of } P_n \\ 2|X| - 1 & \text{if } X \text{ has one end vertex of } P_n \\ 2|X| - 2 & \text{if } X \text{ has both end vertices of } P_n \end{cases}
$$

Proof: Note that this comes from the following simple computation

$$\text{vol}(X) = \sum_{u \in X} \deg_G(u) = \sum_{u \in X} \begin{cases} 2 & \text{if } u \notin \{a, b\} \\ 1 & \text{if } u \in \{a, b\} \end{cases} \tag{6}$$

We will now separate to thee cases as done in the proof of Theorem 24.

Case 1: Suppose $a \notin X$ and $b \notin X$. Then we have

$$\begin{aligned}
\mathrm{vol}(X) &= \sum_{u \in X} \deg_G(u) \\
&= \sum_{u \in X} 2 && \text{(since } u \notin \{a,b\} \text{ and by (6))} \\
&= 2|X|
\end{aligned}$$

Case 2: Without loss of generality, suppose $a \in X$ and $b \notin X$. Then we have

$$\begin{aligned}
\mathrm{vol}(X) &= \sum_{u \in X} \deg_G(u) \\
&= \deg_G(a) + \sum_{u \in X \setminus \{a\}} \deg_G(u) \\
&= 1 + \sum_{u \in X \setminus \{a\}} 2 && \text{(since } u \notin \{a,b\} \text{ and by (6))} \\
&= 1 + 2(|X| - 1) \\
&= 1 + 2|X| - 2 \\
&= 2|X| - 1
\end{aligned}$$

Case 3: Suppose $a, b \in X$. Then we have

$$\begin{aligned}
\mathrm{vol}(X) &= \sum_{u \in X} \deg_G(u) \\
&= \deg_G(a) + \deg_G(b) + \sum_{u \in X \setminus \{a,b\}} \deg_G(u) \\
&= 1 + 1 + \sum_{u \in X \setminus \{a,b\}} 2 && \text{(since } u \notin \{a,b\} \text{ and by (6))} \\
&= 2 + 2(|X| - 2) \\
&= 2 + 2|X| - 4 \\
&= 2|X| - 2
\end{aligned}$$

Since the cases are exhaustive, we know that we have covered every possible make up for $X$ and as such we have computed $\mathrm{vol}(X)$, which matches the statement. Thus we have proven the desired statement. ■

Before we continue to the next statement. It is important for us to note the importance of Theorem 23. Note that if $X$ is chosen such that $\mathrm{vol}(X) \geq |E|$, then by Theorem 23 we have

$$\mathrm{vol}(X) \geq |E| = \mathrm{vol}(X) + \mathrm{vol}(\overline{X}) - |E| \Rightarrow \mathrm{vol}(\overline{X}) \leq |E|$$

And since Theorem 20 tells us that $h_G(X) = h_G(\overline{X})$ we can simply focus on either $X$ or $\overline{X}$ depending on which one satisfies the inequality $\mathrm{vol}(X) \leq |E|$. In other words, the restriction "let $X \in \mathscr{P}^*(V)$ such that $\mathrm{vol}(X) \leq |E|$" is not a restriction, because it can always be done without changing the value of $h_G(X)$.

**Corollary 26:** For all $X \in \mathcal{P}^*(V)$ such that $\mathrm{vol}(X) \leq |E|$, we have the following:

$$h_{P_n}(X) = h(X) = \begin{cases} \frac{|\mathcal{C}(X)|}{|X|} & \text{if } X \text{ has no end vertices of } P_n \\ \frac{2|\mathcal{C}(X)|-1}{2|X|-1} & \text{if } X \text{ has one end vertex of } P_n \\ \frac{|\mathcal{C}(X)|-1}{|X|-1} & \text{if } X \text{ has both end vertices of } P_n \end{cases}$$

Proof: By Theorem 20 we know that we can focus on computing $h(X)$ only, rather than computing both $h(X)$ and $h(\overline{X})$. Furthermore, By Theorem 23 we know that $\mathrm{vol}(X) + \mathrm{vol}(\overline{X}) = 2|E|$. Hence we know that given the value of $\mathrm{vol}(X)$ we can find the value of $\mathrm{vol}(\overline{X})$. Now using Theorems 24 and 25 we arrive at the values shown in the theorem. Note that the min function in the denominator of the definition of $h_G$ is no longer required because we chose our $X$ such that $\mathrm{vol}(X) \leq |E| = \mathrm{vol}(X) + \mathrm{vol}(\overline{X}) - |E|$ which then implies $|E| \leq \mathrm{vol}(\overline{X})$, and hence $\mathrm{vol}(X) \leq \mathrm{vol}(\overline{X})$. ∎

As a fun fact of the above corollary we can find out why the values of the Cheeger constant in our examples had 1 as a common value.

**Corollary 27:** For any $X \in \mathcal{P}^*(V)$, $h(X) = 1$ if and only if $|\mathcal{C}(X)| = |X|$

### 3.4 Final Result

**Theorem 28:** The minimal value of $h_{P_n}(X)$, where $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\}$, is achieved by the case where $X = \{v_1, v_2, \ldots, v_{\lfloor \frac{n}{2} \rfloor}\}$.

Proof: First, let us compare the case where $X$ has no end vertices of $P_n$ and where $X$ has one end vertex in $P_n$. The claim is that the latter case is smaller, so let us look at the opposite. Suppose

$$\frac{|\mathcal{C}(X)|}{|X|} < \frac{2|\mathcal{C}(X)|-1}{2|X|-1}$$

Note that if we simplify this and let $x = |X|$ and $y = |\mathcal{C}(X)|$ then we have

$$\begin{aligned} & \frac{y}{x} < \frac{2y-1}{2x-1} \\ \Leftrightarrow \quad & y(2x-1) < x(2y-1) \\ \Leftrightarrow \quad & 2xy - y < 2xy - x \\ \Leftrightarrow \quad & x < y \\ \Leftrightarrow \quad & |X| < |\mathcal{C}(X)| \end{aligned}$$

However, note that by definition $|\mathcal{C}(X)| \leq |X|$. Hence we know that the above was a contradictory statement. Meaning that for us

$$\frac{|\mathcal{C}(X)|}{|X|} \geq \frac{2|\mathcal{C}(X)|-1}{2|X|-1}$$

Meaning that the case where $X$ has one end vertex is better than the case where $X$ has no end vertices. For the other comparison let us denote

$$\mathcal{X}_1 := \{X \in \mathcal{P}^*(V) : \text{ either } a \in X \text{ or } b \in X\} \qquad \mathcal{X}_2 := \{X \in Pcal^*(V) : a, b \in X\}$$

By Corollary 26 we have

$$\min_{X \in \mathcal{X}_1} (h_G(X)) = \min_{X \in \mathcal{X}_1} \left( \frac{2|\mathcal{C}(X)| - 1}{2|X| - 1} \right)$$

Since the above is trying to compute a minimum we want the numerator to be as small as possible and the denominator to be as large as possible. Note that $|\mathcal{C}(X)| \geq 1$ since it is possible to take $X$ to be the vertex set stated in the statement. Furthermore, we know that we can suppose $\mathrm{vol}(X) \leq |E|$ without any issues. Hence by Theorem 25 we know that $2|X| - 1 \leq |E|$. By the fact that $G$ is the path graph, we know that $|E| = n - 1$. Hence we have $2|X| - 1 \leq n - 1 \Rightarrow |X| \leq \frac{n}{2}$, and since $|X| \in \mathbb{Z}$, it must be the case that $|X| \leq \lfloor \frac{n}{2} \rfloor$. Thus if we combine the two extrema of $|X|$ and $|\mathcal{C}(X)|$ we get the minimal value. In other words

$$\min_{X \in \mathcal{X}_1} (h_G(X)) = \frac{1}{2\lfloor \frac{n}{2} \rfloor - 1} = \begin{cases} \frac{1}{n-1} & \text{if } n \text{ even} \\ \frac{1}{n-2} & \text{if } n \text{ odd} \end{cases}$$

On the other hand by Corollary 26 we have

$$\min_{X \in \mathcal{X}_2} (h_G(X)) = \min_{X \in \mathcal{X}_2} \left( \frac{|\mathcal{C}(X)| - 1}{|X| - 1} \right)$$

Similar to before, we want the numerator to be as small as possible and the denominator to be as large as possible. Note that in this case $|\mathcal{C}(X)| \geq 2$ since for all $X \in \mathcal{X}_2$, $X \subsetneq V$, and if $X$ was a single component then it would connect the two ends of the whole graph together which would violate $X \neq V$. Again, we can assume that $\mathrm{vol}(X) \leq |E| = n - 1$. Thus by Theorem 25 we have $2|X| - 2 \leq n - 1 \Rightarrow 2|X| \leq n + 1$. Thus, because $|X| \in \mathbb{Z}$ we have $|X| \leq \lfloor \frac{n+1}{2} \rfloor$. Thus taking into consideration the extrema of $|\mathcal{C}(X)|$ and $|X|$ we have the following minimal value:

$$\min_{X \in \mathcal{X}_2} (h_G(X)) = \frac{1}{\lfloor \frac{n+1}{2} \rfloor - 1} = \begin{cases} \frac{2}{n-2} & \text{if } n \text{ even} \\ \frac{2}{n-1} & \text{if } n \text{ odd} \end{cases}$$

Let us now compare the two minimal values. Suppose $n$ is even. Then $\min_{X \in \mathcal{X}_1}(h_G(X)) = \frac{1}{n-1}$ and $\min_{X \in \mathcal{X}_2}(h_G(X)) = \frac{2}{n-2}$. Note that we cannot have $n = 2$ due to the fact that if $n$ was equal to 2, then $\mathcal{X}_2$ would be empty since there is only one subset of $V$ which contains both end vertices, and it is not a valid subset for us. Hence we know that $n \geq 4$. Now, suppose in hopes of reaching a contradiction that $\frac{2}{n-2} \leq \frac{1}{n-1}$. This would then have the following sequence of implications:

$$\frac{2}{n-2} \leq \frac{1}{n-1} \Rightarrow 2(n-1) \leq n - 2 \Rightarrow 2n - 2 \leq n - 2 \Rightarrow n \leq 0$$

However, we have already shown how $n \geq 4$. Thus by contradiction we conclude that in the even case,

$$\frac{2}{n-2} > \frac{1}{n-1}$$

In other words

$$\min_{X \in \mathcal{X}_2} (h_G(X)) > \min_{X \in \mathcal{X}_1} (h_G(X))$$

Now, suppose $n$ is odd. In this case we have $\min_{X\in\mathcal{X}_1}(h_G(X)) = \frac{1}{n-2}$ and $\min_{X\in\mathcal{X}_2}(h_G(X)) = \frac{2}{n-1}$. Note that if $n = 3$ then we have $\frac{1}{n-2} = \frac{2}{n-1}$. Now, suppose $n > 3$. Similar to before, suppose in hopes of reaching a contradiction that $\frac{2}{n-1} \leq \frac{1}{n-2}$. By this supposition we have the following implications:

$$\frac{2}{n-1} \leq \frac{1}{n-2} \Rightarrow 2(n-2) \leq n-1 \Rightarrow 2n-4 \leq n-1 \Rightarrow n \leq 3$$

This contradicts our supposition that $n > 3$. Thus we concluded that in the odd case, if $n > 3$

$$\frac{2}{n-1} > \frac{1}{n-2}$$

In other words,

$$\min_{X\in\mathcal{X}_2}(h_G(X)) \geq \min_{X\in\mathcal{X}_1}(h_G(X))$$

with equality holding if an only if $n = 3$. Note that in both the even and odd case we have

$$\min_{X\in\mathcal{X}_2}(h_G(X)) \geq \min_{X\in\mathcal{X}_1}(h_G(X))$$

with equality holding only when $n = 3$. Thus we can definitely state that when $X$ contains only one end-vertex of $P_n$, is a single connected component, and has size $\lfloor \frac{n}{2} \rfloor$, then the minimal value of $h_G(X)$ is achieved when ranging over all possible $X$. In other words if we let $P_n = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $E = \{\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}\}$, then if $X = \{v_1, \ldots, v_{\lfloor \frac{n}{2} \rfloor}\}$ then $h_{P_n} = h_{P_n}(X)$. This is precisely the statement we wished to prove.

$\blacksquare$