Path Finding Markov Chain

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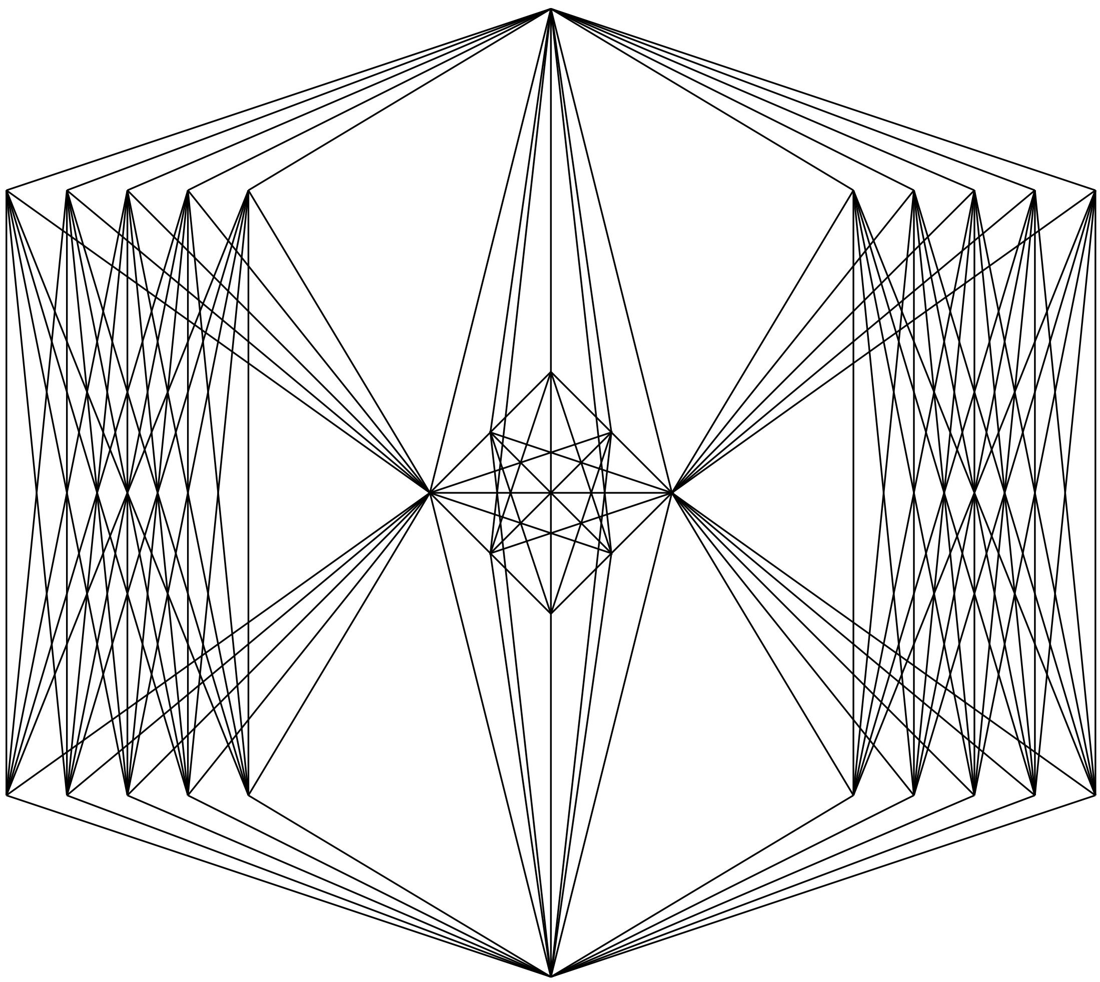


1 Introduction

This work is largely built off of the work of [MP15]. The goal of their paper was to build a Markov chain on st-paths on a given graph G, in order to find the shortest path from vertex s to vertex t. They cite the fact that the shortest st-path problem is computationally difficult, and hence they wish to find an algorithm for finding the shortest st-path using a probabilistic approach.

We analyzed their work and use a computational and graph theoretic approach to better their algorithm. [MP15] introduced two distinct Markov chains. While both focused on st-paths on planar graphs, the first is applicable to any planar graph and the second only works for 2-D lattice graphs. They only make concrete statements about the second Markov chain while conjecturing about the first. We explore both to see if we can draw any conclusions of our own.

Furthermore, an important observation about both Markov chains, is that the probability of going from one path to another depends on the planar embedding that is used. We wish to explore this further to see if there is an optimal embedding that should be used.

2 Graph Theory Definitions

Definition 1: A graph is a collection of vertices (nodes) and edges (lines between vertices).

Definition 2: A simple graph is a graph where the edges are incident to exactly two distinct vertices and are non-directional.

Definition 3: A directed graph is a graph where edges are incident to exactly two (not necessarily distinct) vertices and are directional.

The distinctions made above on the type of graph that we are interacting with is very important. In order to go from one type of graph to another while maintaining some form of graphical topology we introduce the following function:

Definition 4: The simplification function denoted Υ takes a directed graph, keeps the vertices the same, but then turns all directed edges to non-directed edges (if multi-edges occur because of this, it keeps only one edge out of the lot) and removes any loops present.

Definition 5: A homomorphism from graph $G = (V,E)$ to graph $H = (V',E')$ is a tuple of functions $h = (h_V, h_E)$ where $h_V : V \to V'$ and $h_E : E \to E'$ such that if $\{a,b\} \in E$ then $h_E(\{a,b\}) = \{h_V(a), h_v(b)\} \in E'$.

Definition 6: We call a graph homomorphism $h = (h_V, h_E)$ a graph isomorphism if h_V and h_E are bijections.

Definition 7: Let $G = (V,E)$ be a graph. A sequence of vertices $\{v_1, v_2, \ldots, v_n\}$ such that for all $1 \leq i < n$, $\{v_i, v_{i+1}\} \in E$, is called a walk in G starting at v_1 and ending in v_n.

Definition 8: Let $\mathfrak{x} = \{v_1, \ldots, v_n\}$ be a walk in the graph G. We define the length of $\mathfrak{x}$ to be $|x| := n-1$.

Definition 9: Let $G = (V,E)$ be a graph. A path is a walk $\{v_1, \ldots, v_n\}$ such that $v_i \neq v_j$ for all $1 \leq i \neq j \leq n$.

Definition 10: Let $\mathfrak{x}$ be a path in G. If $\mathfrak{x}$ starts with the vertex s and ends with the vertex t, then we call $\mathfrak{x}$ an st-path.

For the rest of this section, suppose $G = (V,E)$ is a simple graph.

Definition 11: G is connected if for every $u, v \in V$ there exists a uv-path in G. On the other hand, a graph is disconnected if it is not connected.

Definition 12: For a vertex $v \in V$ we define vertex deletion to be the action of making the new graph $G - v := (V', E')$ where $V' := V \setminus \{v\}$ and $E' := E \setminus \{\{u,w\} \in E : u \neq v \text{ and } w \neq v\}$. Similarly we define vertex set deletion by $S \subseteq V$ to be the action of deleting all the vertices of S from G. This new graph is denoted by $G - S$.

Definition 13: A proper subset $S \subseteq V$ is a (vertex) cut set if $G - S$ is a disconnected graph.

Definition 14: The connectivity of G, denoted by $\kappa(G)$, is the cardinality of the minimal cut set of G.

Definition 15: For $k \in \mathbb{N}$ we say that G is k-connected if $k \leq \kappa(G)$.

3 Spherical Topology

For our purposes we need to be able to talk about points in $\mathbb{R}^3$ in terms of spherical coordinates. It is well known that any point in $\mathbb{R}^3$ can be represented by the cartesian coordinates $x, y, z \in \mathbb{R}$, and any point in $\mathbb{R}^3$ can also be represented by the spherical coordinates θ, ϕ and r. Where $\theta \in [0, \pi]$ is the the angle between the positive z-axis, the origin, and our point, $\phi \in [0, 2\pi)$ is the counter-clockwise rotation angle from the positive x-axis, the origin, and our points projection onto the $z = 0$ plane, and $r \in \mathbb{R}_{\geq 0}$ is the distance from the origin to our point.

Even though this method of substitution was sufficient for calculus, it will not be sufficient for our purposes. This is due to the fact that there is no bijection between $\mathbb{R}^3$ and $[0, \pi] \times [0, 2\pi) \times \mathbb{R}_{\geq 0}$, namely at the origin and along the z-axis there is an infinite number of ways to represent our points in $\mathbb{R}^3$ via spherical coordinates.

Definition 16: We define the spherical coordinate region as

$$\mathscr{R} := (0, \pi) \times [0, 2\pi) \times \mathbb{R}_{>0} \cup \{0\} \times \{0\} \times \mathbb{R}_{\geq 0} \cup \{\pi\} \times \{0\} \mathbb{R}_{>0} \subseteq \mathbb{R}^3$$

We will show that $\mathscr{R}$ will identify a one-to-one correspondence with $\mathbb{R}^3$ through the spherical coordinates that we like to think.

Remark 17: Even though we will show that $\mathscr{R}$ will identify spherical coordinates and $\mathbb{R}^3$ in a bijective way, $\mathscr{R}$ is not the only subset that does so.

Theorem 18: There exists a bijection between $\mathbb{R}^3$ and $\mathscr{R}$.

Proof: As mentioned above, it is well known that the function

$$f : [0, \pi] \times [0, 2\pi) \times \mathbb{R}_{\geq 0} \to \mathbb{R}^3$$

such that

$$(\theta, \phi, r) \mapsto (r \sin(\theta) \cos(\phi), r \sin(\theta) \sin(\phi), r \cos(\theta))$$

is the correspondence between polar and cartesian coordinates. Hence we know that f is a surjective function. However it is not injective across the z-axis. In particular, we know that

(a) $\forall\, \theta_1, \theta_2 \in [0, \pi], \phi_1, \phi_2 \in [0, 2\pi)$, we have

$$f(\theta_1, \phi_1, 0) = (0, 0, 0) = f(\theta_2, \phi_2, 0)$$

(b) $\forall\, \phi_1, \phi_2 \in [0, 2\pi), r \in [0, \infty)$ we have

$$f(0, \phi_1, r) = (0, 0, r) = f(0, \phi_2, r)$$

(c) $\forall\, \phi_1, \phi_2 \in [0, 2\pi), r \in [0, \infty)$ we have

$$f(\pi, \phi_1, r) = (0, 0, -r) = f(\pi, \phi_2, r)$$

Furthermore, we know that these are the only non-injective points for our function f. Hence, if we restrict our domain, we can get a bijective function. Note that on $(0,\pi)\times[0,2\pi)\times(0,\infty)$, f is injective. Now, we know that for $r\in(0,\infty)$, the preimage of $(0,0,r)$ is $\{0\}\times[0,2\pi)\times\{r\}$. As such, for all $r\in(0,\infty)$, we restrict the domain of f such that the pre-image of $(0,0,r)$ is only $\{0\}\times\{0\}\times\{r\}$. Hence, our new function is no longer defined on $\{0\}\times(0,2\pi)\times\{r\}$. Doing a similar action for the pre-image of the vector $(0,0,-r)$, we see that our function is no longer defined on $\{\pi\}\times(0,2\pi)\times\{r\}$ for all $r\in(0,\infty)$. Lastly, note that the pre-image of the vector $(0,0,0)$ is $[0,\pi]\times[0,2\pi)\times\{0\}$. Again, we restrict our new function's domain such that $(0,0,0)$ only has preimage $\{0\}\times\{0\}\times\{0\}$. Hence our new function is not defined on $(0,\pi]\times(0,2\pi)\times\{0\}$. Note that we have made all the points along the z-axis have a singleton as a pre-image, making the new function injective on its full domain. Thus, we conclude that the function $g:\mathscr{R}\to\mathbb{R}^3$ such that

$$(\theta,\phi,r)\mapsto(r\sin(\theta)\cos(\phi),r\sin(\theta)\sin(\phi),r\cos(\theta))$$

is a bijective function between $\mathscr{R}$ and $\mathbb{R}^3$. ■

Definition 19: We define the functions $A:\mathbb{R}^3\to[0,\pi]$, $B:\mathbb{R}^3\to[0,2\pi)$, and $C:\mathbb{R}^3\to\mathbb{R}_{\geq 0}$ such that

$$A(x,y,z)=\begin{cases}\arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) & \text{if } z>0\\ \pi+\arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) & \text{if } z<0\\ \frac{\pi}{2} & \text{if } z=0, x\neq 0 \text{ and } y\neq 0\\ 0 & \text{if } x=y=z=0\end{cases}$$

$$B(x,y,z)=\begin{cases}\arctan\left(\frac{y}{x}\right) & \text{if } x>0\\ \arctan\left(\frac{y}{x}\right)+\pi & \text{if } x<0 \text{ and } y\geq 0\\ \arctan\left(\frac{y}{x}\right)-\pi & \text{if } x<0 \text{ and } y<0\\ \frac{\pi}{2} & \text{if } x=0 \text{ and } y>0\\ \frac{3\pi}{2} & \text{if } x=0 \text{ and } y<0\\ 0 & \text{if } x=y=0\end{cases}$$

$$C(x,y,z)=\sqrt{x^2+y^2+z^2}$$

Theorem 20: The function from $\mathbb{R}^3\to\mathscr{R}$ such that $(x,y,z)\mapsto(A(x,y,z),B(x,y,z),C(x,y,z))$ is g^{-1}.

Proof: By construction. ■

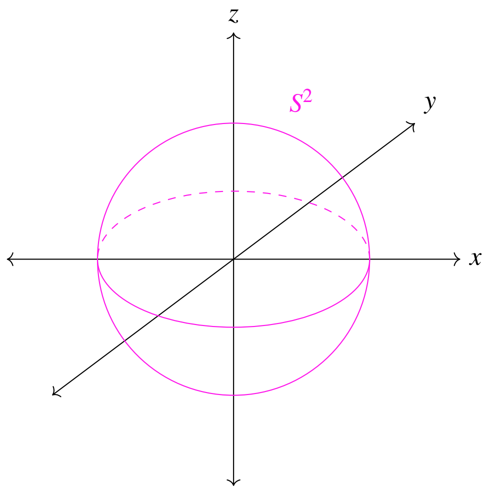
Definition 21: We denote the unital spherical coordinate region by

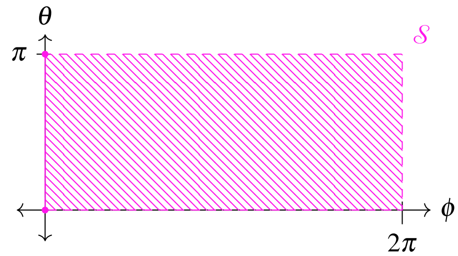
$$\mathscr{S}:=((0,\pi)\times(0,2\pi))\cup([0,\pi]\times\{0\})\subseteq\mathbb{R}^2$$

We also denote the unit sphere in $\mathbb{R}^3$ by S^2. Lastly, for both sets we donte

$$\mathscr{S}_\times:=\mathscr{S}\setminus\{(0,0)\}\qquad\qquad S^2_\times:=S^2\setminus\{(0,0,1)\}$$

Remark 22: The function g implies a bijection between $\mathscr{S}$ and S^2, where each r input for g has value 1. By abuse of notation we denote the function from $\mathscr{S}$ to S^2 such that $(\theta,\phi)\mapsto g(\theta,\phi,1)=(\sin(\theta)\cos(\phi),\sin(\theta)\sin(\phi),\cos(\theta))$ by g.





Note that we have now found a way to talk about points in S^2 by only two variables. Even more, note that there is an inherent topology on S^2. But this topology is different than the Euclidean topology of $\mathbb{R}^3$. Let us try to be more precise about this.

Definition 23: We define the function $d: \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ such that

$$d((\theta_1, \phi_1), (\theta_2, \phi_2)) = \arccos(\sin(\theta_1)\sin(\theta_2)\cos(\phi_1 - \phi_2) + \cos(\theta_1)\cos(\theta_2))$$

Definition 24: We define the function $\tilde{d}: S^2 \to S^2 \to \mathbb{R}$ such that

$$\tilde{d} = \arccos(\vec{v} \cdot \vec{w}) = \arccos(\langle \vec{v}, \vec{w} \rangle)$$

where $\langle \, , \, \rangle$ is the Euclidean inner product (i.e. the dot product).

The reason we define d and $\tilde{d}$ is because they both are defining a notion of distance on the unit sphere by walking along the surface of the sphere. Where $\tilde{d}$ works with points directly in the third dimension, d is the simpler version of it which only deals with two variables. We can be more precise about the connection between d and $\tilde{d}$ via the following lemma.

Lemma 25: For all $\vec{v}, \vec{w} \in \mathcal{S}$, $d(\vec{v}, \vec{w}) = \tilde{d}(g(\vec{v}, \vec{w}))$.

Proof: Let $(\theta_1, \phi_1), (\theta_2, \phi_2) \in \mathcal{S}$ and consider the following:

$$\begin{aligned}
\tilde{d}(g(\theta_1, \phi_1), g(\theta_2, \phi_2)) &= \tilde{d}\left(\begin{bmatrix} \sin(\theta_1)\cos(\phi_1) \\ \sin(\theta_1)\sin(\phi_1) \\ \cos(\theta_1) \end{bmatrix}, \begin{bmatrix} \sin(\theta_2)\cos(\phi_2) \\ \sin(\theta_2)\sin(\phi_2) \\ \cos(\theta_2) \end{bmatrix} \right) \\
&= \arccos(\sin(\theta_1)\cos(\phi_1)\sin(\theta_2)\cos(\phi_2) + \sin(\theta_1)\sin(\phi_1)\sin(\theta_2)\sin(\phi_2) + \cos(\theta_1)\cos(\theta_2)) \\
&= \arccos(\sin(\theta_1)\sin(\theta_2)(\cos(\phi_1)\cos(\phi_2) + \sin(\phi_1)\sin(\phi_2)) + \cos(\theta_1)\cos(\theta_2)) \\
&= \arccos(\sin(\theta_1)\sin(\theta_2)\cos(\phi_1 - \phi_2) + \cos(\theta_1)\cos(\theta_2)) \\
&= d((\theta_1, \phi_1), (\theta_2, \phi_2))
\end{aligned}$$

■

Remark 26: Thus, since g is a bijection, we know that for all $\vec{v}, \vec{w} \in S^2, d(g^{-1}(\vec{v}), g^{-1}(\vec{w})) = \tilde{d}(\vec{v}, \vec{w})$. In other words, if we consider $\mathcal{S}$ and S to be the same object, $d = \tilde{d}$.

Theorem 27: d is a metric on $\mathcal{S}$.

Proof: Let $\vec{v} = (\theta_1, \phi_1), \vec{w} = (\theta_2, \phi_2) \in \mathcal{S}$ and consider $d(\vec{v}, \vec{w})$. By definition we know that $d(\vec{v}, \vec{w}) \in \mathtt{Im}(\arccos) = [0, \pi]$. Hence we know that $d(\vec{v}, \vec{w}) \geq 0$. Furthermore, note that we have the following:

$$\begin{aligned} d(\vec{v}, \vec{w}) &= \tilde{d}(g(\vec{v}), g(\vec{w})) && \text{(by Lemma 25)} \\ &= \arccos(g(\vec{v}) \cdot g(\vec{w})) \\ &= \arccos(g(\vec{w}) \cdot g(\vec{v})) \\ &= \tilde{d}(g(\vec{w}), g(\vec{v})) \\ &= d(\vec{w}, \vec{v}) \end{aligned}$$

Hence we know that d is symmetric.

By abuse of notation let $\vec{v} \cdot \vec{w} := \cos(d(\vec{v}, \vec{w}))$.

Now, suppose that $\vec{v} \neq \vec{w}$. We separate this supposition into three cases:

Case 1: Suppose $\theta_1 = \theta_2$ and $\phi_1 \neq \phi_2$. Then we have

$$\begin{aligned} \vec{v} \cdot \vec{w} &= \sin(\theta_1)\sin(\theta_2)\cos(\phi_1 - \phi_2) + \cos(\theta_1)\cos(\theta_2) \\ &= \sin^2(\theta_1)\cos(\phi_1 - \phi_2) + \cos^2(\theta_1) \\ &= \sin^2(\theta_1)\cos(\phi_1 - \phi_2) + 1 - \sin^2(\theta_1) \\ &= \sin^2(\theta_1)(\cos(\phi_1 - \phi_2) - 1) + 1 \end{aligned}$$

Since $\theta_1 \in [0, \pi]$ we know that $0 \leq \sin^2(\theta_1) \leq 1$. As such we know that $\vec{v} \cdot \vec{w} \leq \cos(\phi_1 - \phi_2)$. Now, since $\phi_1, \phi_2 \in [0, 2\pi)$, we know that $\cos(\phi_1 - \phi_2) = 1$ if and only if $\phi_1 = \phi_2$ which is not allowed by the definition of $\mathcal{S}$. Hence $\vec{v} \cdots \vec{w} < 1$. Consequently, from this inequality and the fact that arccos is a monotinically decreasing function, we conclude that $d(\vec{v}, \vec{w}) = \arccos(\vec{v} \cdot \vec{w}) > \arccos(1) = 0$.

Case 2: Suppose $\theta_1 \neq \theta_2$ and $\phi_1 = \phi_2$. Note that we have

$$\begin{aligned} \vec{v} \cdots \vec{w} &= \sin(\theta_1)\sin(\theta_2)\cos(\phi_1 - \phi_2) + \cos(\theta_1)\cos(\theta_2) \\ &= \sin(\theta_1)\sin(\theta_2)\cos(0) + \cos(\theta_1)\cos(\theta_2) \\ &= \sin(\theta_1)\sin(\theta_2) + \cos(\theta_1)\cos(\theta_2) \\ &= \cos(\theta_1 - \theta_2) \end{aligned}$$

By similar reasoning as in Case 1 we conclude that $\vec{v} \cdot \vec{w} < 1$, which implies $d(\vec{v}, \vec{w}) > 0$.

Case 3: Suppose $\theta_1 \neq \theta_2$ and $\phi_1 \neq \phi_2$. Note that by the reasons in Case 1 we know that $\cos(\phi_1 - \phi_2) < 1$. As such we know that

$$\vec{v} \cdot \vec{w} < \sin(\theta_1)\sin(\theta_2) + \cos(\theta_1)\cos(\theta_2) = \cos(\theta_1 - \theta_2)$$

Again, we know that $\cos(\theta_1 - \theta_2) < 1$ as such, we also know that $d(\vec{v}, \vec{w}) > 0$

Note that we have shown that if $\vec{v} \neq \vec{w}$, then $d(\vec{v}, \vec{w}) > 0$.

Now, all that is left to show is that d satisfies the triangle inequality. Let $\vec{u} = (\theta_1, \phi_1), \vec{v} = (\theta_2, \phi_2), \vec{w} = (\theta_3, \phi_3)$, $\vec{x} = g(\vec{u}), \vec{y} = g(\vec{v}), \vec{z} = g(\vec{w}) \in S^2 \subseteq \mathbb{R}^3$, and finally, let $d_{xy} = \tilde{d}(\vec{x}, \vec{y}), d_{yz} = \tilde{d}(\vec{y}, \vec{z}), d_{xz} = \tilde{d}(\vec{x}, \vec{z})$. Note that by definition we know that

$$\cos(d_{xy}) = \langle \vec{x}, \vec{y} \rangle, \qquad \cos(d_{yz}) = \langle \vec{y}, \vec{z} \rangle, \qquad \cos(d_{xz}) = \langle \vec{x}, \vec{z} \rangle$$

where $\langle \, , \, \rangle$ is the Euclidean inner product.

Now, let $\vec{q}$ be the normalized vector of the othogonal component of $\vec{x}$ onto $\vec{y}$. In other words, let

$$\vec{q} = \frac{1}{||\vec{x} - P_{\vec{y}}(\vec{x})||} \left(\vec{x} - P_{\vec{y}}(\vec{x}) \right)$$

Note that we have:

$$P_{\vec{y}}(\vec{x}) = \frac{\langle \vec{y} \vec{x} \rangle}{\langle \vec{y}, \vec{y} \rangle} \vec{y} = \frac{\cos(d_{xy})}{\cos(\tilde{d}(\vec{y}, \vec{y}))} \vec{y} = \frac{\cos(d_{xy})}{\cos(0)} \vec{y} = \cos(d_{xy}) \vec{y}$$

Furthermore, note that we have the following:

$$\begin{aligned}
\|\vec{x} - P_{\vec{y}}(\vec{x})\| &= \|\vec{x} - \cos(d_{xy})\vec{y}\| \\
&= \sqrt{\langle \vec{x} - \cos(d_{xy})\vec{y}, \vec{x} - \cos(d_{xy})\vec{y} \rangle} \\
&= \sqrt{\langle \vec{x}, \vec{x} \rangle - \cos(d_{xy})\langle \vec{x}, \vec{y} \rangle - \cos(d_{xy})\langle \vec{y}, \vec{x} \rangle + \cos^2(d_{xy})\langle \vec{y}, \vec{y} \rangle} \\
&= \sqrt{1 - \cos(d_{xy})\langle \vec{x}, \vec{y} \rangle - \cos(d_{xy})\langle \vec{x}, \vec{y} \rangle + \cos^2(d_{xy})} \\
&= \sqrt{1 - \cos(d_{xy})\cos(d_{xy}) - \cos(d_{xy})\cos(d_{xy}) + \cos^2(d_{xy})} \\
&= \sqrt{1 - \cos^2(d_{xy})} \\
&= \sin(d_{xy})
\end{aligned}$$

Thus we conclude that

$$\vec{q} = \frac{1}{\sin(d_{xy})}(\vec{x} - \cos(d_{xy})\vec{y}) \Rightarrow \vec{x} = \sin(d_{xy})\vec{q} + \cos(d_{xy})\vec{y}$$

By replacing $\vec{x}$ with $\vec{z}$ we also arrive at

$$\vec{z} = \sin(d_{yz})\vec{r} + \cos(d_{yz})\vec{y}$$

where $\vec{r}$ is the normalized vector of the orthogonal component of $\vec{z}$ onto $\vec{y}$.

It is important to note that since $\vec{q}$ and $\vec{r}$ are normalized vectors, we know that $\vec{q}, \vec{r} \in S^2$. Hence we know that $\langle \vec{q}, \vec{r} \rangle = \cos(\tilde{d}(\vec{q}, \vec{r})) \geq -1$.

With this information we are ready to take a look at

$$d(\vec{u}, \vec{w}) = \tilde{d}(\vec{x}, \vec{z}) = \arccos(\langle \vec{x}, \vec{z} \rangle)$$

Let us focus on the inner product. Note that we have the following:

$$\begin{aligned}
\langle \vec{x}, \vec{z} \rangle &= \langle \sin(d_{xy})\vec{q} + \cos(d_{xy})\vec{y}, \sin(d_{yz})\vec{r} + \cos(d_{yz})\vec{y} \rangle \\
&= \sin(d_{xy})\sin(d_{yz})\langle \vec{q}, \vec{r} \rangle + \sin(d_{zy}\cos(d_{yz})\langle \vec{q}, \vec{y} \rangle + \cos(d_{xy})\sin(d_{yz})\langle \vec{y}, \vec{r} \rangle + \cos(d_{xy})\cos(d_{yz})\langle \vec{y}, \vec{y} \rangle \\
&= \sin(d_{xy})\sin(d_{yz})\langle \vec{q}, \vec{r} \rangle + \cos(d_{xy})\cos(d_{yz})\langle \vec{y}, \vec{y} \rangle \\
&= \sin(d_{xy})\sin(d_{yz})\langle \vec{q}, \vec{r} \rangle + \cos(d_{xy})\cos(d_{yz}) \\
&\geq \cos(d_{xy})\cos(d_{yz}) - \sin(d_{xy})\sin(d_{yz}) \\
&= \cos(d_{xy} + d_{yz})
\end{aligned}$$

Note that if $d_{xy} + d_{yz} \leq \pi$, then $\arccos(\cos(d_{xy} + d_{yz})) = d_{xy} + d_{yz}$. As such we would have $d(\vec{u}, \vec{w}) \leq \arccos(\cos(d_{xy} + d_{yz})) = d_{xy} + d_{yz} = d(\vec{u}, \vec{v}) + d(\vec{v}, \vec{w})$ which is what we want.

On the other hand, if $d_{xy} + d_{yz} > \pi$, then we have $d(\vec{u}, \vec{w}) \leq \pi < d_{xy} + d_{yz} = d(\vec{u}, \vec{v}) + d(\vec{v}, \vec{w})$ which is what we want.

Thus, we conclude that d satisfies the triangle inequality. Hence, we know that d is a metric on $\mathcal{S}$. ■

Corollary 28: $\tilde{d}$ is a metric on S^2.

Proof: This follows from the fact that through the function g, d and $\tilde{d}$ commute on $\mathcal{S}$ and S^2.

$$\begin{tikzcd}
\mathcal{S} \arrow[rr, "g"] \arrow[dr, "d"'] & & S^2 \arrow[dl, "\tilde{d}"] \\
& \mathbb{R}_{\geq 0} &
\end{tikzcd}$$

■

Now that we have a metric on $\mathcal{S}$ and S^2 we can start talking about the topology of the unit sphere. In particular the induced topology by the metric d.

The last idea that we will cover in this section is the idea of projecting a sphere onto a plane. This projection is not a linear projection, but rather the projection of the (unit) sphere onto the plane by removing the topmost point, and associating points on the sphere with points on the plane via the line that passes through the original point and the topmost point of the sphere.

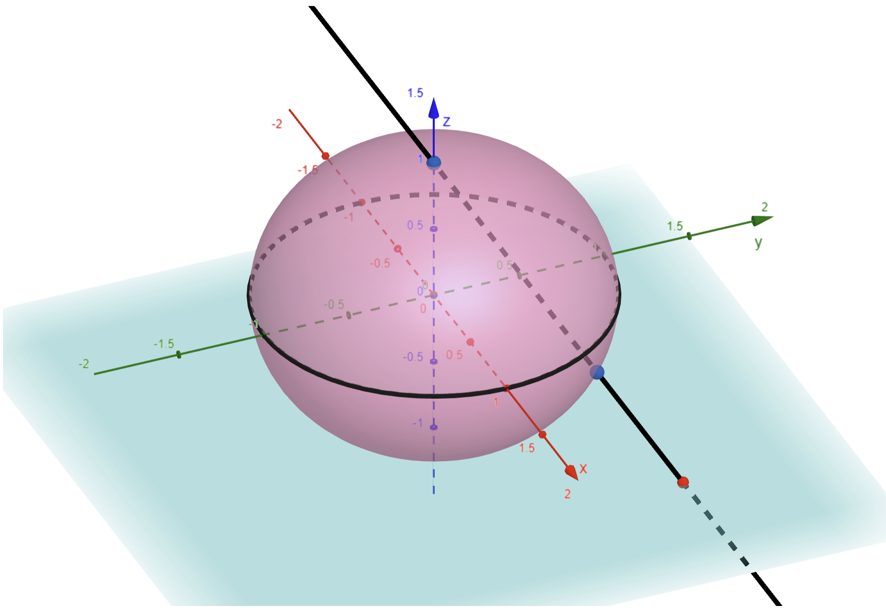


Figure 1: This figure is showing S^2 in red along with its intersection with the $z=0$ plane, and the $z=-1$ plane in cyan. The figure also shows how the bottom blue point is getting projected onto the $z=-1$ plane via the line going through the point and the top of S^2.

Definition 29: We define $p:\mathcal{S}_\times \to \mathbb{R}^2$ such that for $\vec{v}=(\theta,\phi)\in\mathcal{S}_\times$

$$p(\vec{v})=\begin{bmatrix}\frac{-2\sin(\theta)\cos(\phi)}{\cos(\theta)-1}\\ \frac{-2\sin(\theta)\sin(\phi)}{\cos(\theta)-1}\end{bmatrix}=\frac{-2}{\cos(\theta)-1}\begin{bmatrix}\sin(\theta)\cos(\phi)\\ \sin(\theta)\sin(\phi)\end{bmatrix}$$

Remark 30: Note that

$$p(\vec{v})=P_{\{\vec{e}_1,\vec{e}_2\}}(g(\vec{v}))$$

where $P_{\{\vec{e}_1,\vec{e}_2\}}$ is the projection map onto $\text{span}\{\vec{e}_1,\vec{e}_2\}$.

Definition 31: We define $\tilde{p}:S^2_\times\to\mathbb{P}:=\{(a,b,-1)\in\mathbb{R}^3|a,b\in\mathbb{R}\}$ such that for $z=g(\theta,\phi)\in S^2_\times$ we have

$$\begin{bmatrix}\sin(\theta)\cos(\phi)\\ \sin(\theta)\sin(\phi)\\ \cos(\theta)\end{bmatrix}\mapsto\begin{bmatrix}\frac{-2\sin(\theta)\cos(\phi)}{\cos(\theta)-1}\\ \frac{-2\sin(\theta)\sin(\phi)}{\cos(\theta)-1}\\ -1\end{bmatrix}$$

Remark 32: Note that if $\vec{v}=(\theta,\phi)=g^{-1}(\vec{z})$ for $\vec{z}\in S^2_\times$, then $p(\vec{v})=P_{\{\vec{e}_1,\vec{e}_2\}}(\tilde{p}(\vec{z}))$. As such we have the following commuting diagram:

$$\begin{tikzcd} S^2_\times \arrow[d, "\tilde{p}"'] & \mathcal{S}_\times \arrow[l, "g"'] \arrow[d, "p"] \\ \mathbb{P} \arrow[r, "P_{\{\vec{e}_1,\vec{e}_2\}}"'] & \mathbb{R}^2 \end{tikzcd}$$

and note that in this case $\mathbb{P}$ and $\mathbb{R}^2$ have a 1-to-1 correspondence through $P_{\{\vec{e}_1,\vec{e}_2\}}$ where $P^{-1}_{\{\vec{e}_1,\vec{e}_2\}} : \mathbb{R}^2 \to \mathbb{P}$ is definied such that $(x,y) \mapsto (x,y,-1)$.

Definition 33: For a given point $\vec{v} = (\theta,\phi) \in \mathcal{S}$ we define the $\vec{v}$ rotation as the function, $R_{\vec{v}}$ which takes S^2 and rotates it such that $\vec{v}$ becomes the new top of S^2. Explicitly, $R_{\vec{v}} : \mathcal{S} \to \mathcal{S}$ such that $R_{\vec{v}}(\vec{w}) = g^{-1}(R_y R_z g(\vec{w}))$ where

$$R_z = \begin{bmatrix} \cos(-\phi) & -\sin(-\phi) & 0 \\ \sin(-\phi) & \cos(-\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad R_y = \begin{bmatrix} \cos(-\theta) & 0 & \sin(-\theta) \\ 0 & 1 & 0 \\ -\sin(-\theta) & 0 & \cos(-\theta) \end{bmatrix}$$

Remark 34: By simple computations, if $\vec{v} = (\theta_1,\phi_1) \in \mathcal{S}$ and $\vec{w} = (\theta_2,\phi_2) \in \mathcal{S}$ we know have the following:

$$R_y R_z g(\vec{w}) = \begin{bmatrix} \cos(\theta_1)\sin(\theta_2)\cos(\phi_2-\phi_1) - \sin(\theta_1)\cos(\theta_2) \\ \sin(\theta_2)\sin(\phi_2-\phi_1) \\ \sin(\theta_1)\sin(\theta_2)\cos(\phi_2-\phi_1) + \cos(\theta_1)\cos(\theta_2) \end{bmatrix}$$

Definition 35: We define the projection through $\vec{v}$ of the unital sphere to the plane to be the function $\mathfrak{P}_{\vec{v}} = p \circ R_{\vec{v}}$.

4 Abstract Graph Theory

The key ideas behind this and the next section come from the lecture notes [Eri23]. With the understanding of both basic graph theory, spherical topology, and the connection between spherical topology and $\mathbb{R}^2$, we can now approach abstract graphs. After we have done so, we will have all the necessary tools to talk about graph embeddings and how to find all distinct graph embeddings.

Definition 36: We define an abstract graph to be a quadruple $(V,D,\texttt{rev},\texttt{head})$, where

- V is a non-empty set. Elements of this set are called vertices.
- D is a non-empty set. Elements of this set are called darts.
- $\texttt{rev} : D \to D$ is an involution with no stationary points (i.e. $\texttt{rev}(\texttt{rev}(d)) = d \neq \texttt{rev}(d)$ for all $d \in D$). We call $\texttt{rev}$ the reversal function of the abstract graph. Sometimes we denote $\texttt{rev}(d)$ by $-d$.
- $\texttt{head} : D \to V$ is function. We call this function the head function of the abstract graph.

We denote an abstract graph typically by a capital letter, mainly the letter G.

Definition 37: Given an abstract graph $G = (V,D,\texttt{rev},\texttt{head})$ we define the tail function to be $\texttt{tail} : D \to V$ such that $\texttt{tail} = \texttt{head} \circ \texttt{rev}$.

Definition 38: For an abstract graph $G = (V,D,\texttt{rev},\texttt{head})$ we define an edge of G to be an unordered set of the form $\{d,\texttt{rev}(d)\}$ for some $d \in D$. We denote the set of edges of G by E or by $E(G)$ when the context requires it.

For a particular dart $d \in D$ we define the associated edge of d to be $|d| := \{d,\texttt{rev}(d)\}$.

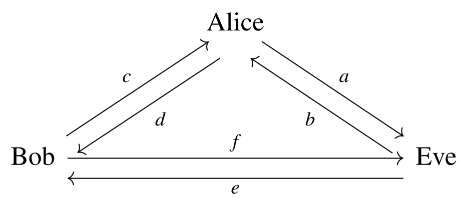
For a particular edge $e \in E$ we denote the edge by arbitrary choosing one of its darts to be e^- and the other to be e^+ (i.e. $e = \{e^-,e^+\}$ where $e^-,e^+ \in D$ and $e^+ = \texttt{rev}(e^-)$).

Remark 39: Note that we may simply refer to the abstract graph via its vertices and edges instead of the vertices, darts, reverse function, and head function. Furthermore, note that by considering this, we also get the definition of a graph homomorphism and graph isomorphism on abstract graphs. But it will not be explicitly stated here.

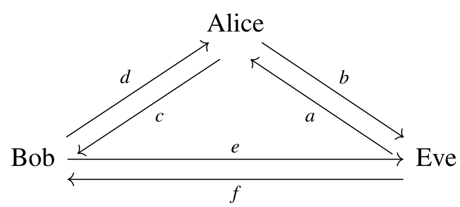
Example 40: Consider the abstract graph $G = (V, D, \texttt{rev}, \texttt{head})$ where

- $V = \{$Alice, Bob, Eve$\}$
- $D = \{a, b, c, d, e, f\}$
- $\texttt{rev} : D \to D$ such that $a \mapsto b$, $c \mapsto d$, $e \mapsto f$
- $\texttt{head} : D \to V$ such that $a \mapsto$ Alice, $b \mapsto$ Eve $c \mapsto$ Bob, $d \mapsto$ Alice, $e \mapsto$ Eve, $f \mapsto$ Bob.

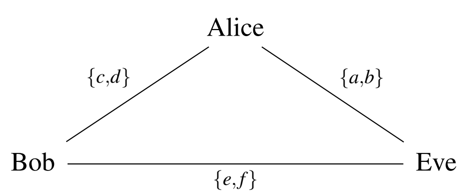
Note that keeping track of darts we have defined the following graph:



However, since it is not specified in our interpretation that `head` refers to where the directed edge starts or terminates. Hence we also have the interpretation:



That is why we have the simple interpretation of the graph G as being:



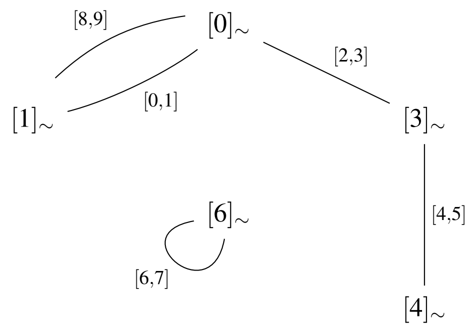
Definition 41: We define a topological graph to be the quotient of disjoint unions of closed intervals by an equivalence relation of the end points. We denote this space typically by G^T

Example 42: We can consider the disjoint unions $[0,1] \sqcup [2,3] \sqcup [4,5] \sqcup [6,7] \sqcup [8,9]$ and our equivalence relation relating $0 \sim 2$, $3 \sim 5$, $6 \sim 7$, $0 \sim 8$, $1 \sim 9$ extended in an equivalence way. Denote this quotient space by G^T.

In this case we see that to us $0 \equiv 2 \equiv 8$, $1 \equiv 9$, $3 \equiv 5$, $6 \equiv 7$. Hence we have five vertices $[0]_\sim, [1]_\sim, [3]_\sim, [4]_\sim, [6]_\sim$. These will be our vertices.

The edges of the graph are the intervals in the disjoint union, but now with the equivalence relation of the end points. In this case our edges are the intervals $[0,1], [2,3], [4,5], [6,7], [8,9]$.

Note that the topological space that we have made is no longer Hausdorff due to the fact that $0, 2 \in G^T$ and $0 \neq 2$, however these two distinct points are not disjoint by neighborhoods. Either way the graph that we have made is the following:



Remark 43: Note that both an abstract graph and a topological graph are not graphs in the sense that we discussed in Section 2, but rather an abstract object which can be interpreted as a graph. In our interpretations we do not exclude the possibility of multi-edges or loops, but we will not be working with them much for the purposes of this paper. Furthermore, since there is this distinction between darts and edges, when we are looking at a graph made up of vertices and edges, it is automatically undirected.

Remark 44: It should be clear in the fact that we can turn any abstract graph into a topological graph and vice versa.

Now that we have the abstract and topological ways to represent a graph, we would like to find a way to represent graphs visually.

Definition 45: Let $G^T = \bigsqcup_i [n_i, n_{i+1}]/\sim$ be a topological graph. Let V be the endpoints of the intervals $\{[n_i, n_{i+1}]\}$ under the equivalence $\sim$. Furthermore, let E be the set of distinct intervals. An embedding function of G^T onto the plane is a tuple of functions $\mathcal{E} = (\mathcal{E}_V, \mathcal{E}_E)$ where

$$\mathcal{E}_V : V \longrightarrow \mathbb{R}^2 \qquad \text{and} \qquad \mathcal{E}_E : E \longrightarrow \mathcal{P}\left(\mathbb{R}^2\right)$$

where $\mathcal{P}\left(\mathbb{R}^2\right)$ is the power set, such that for any $v \in V$, $\mathcal{E}_V(v)$ is a point in the plane, and if $[u,w] \in E$ then $\mathcal{E}_E([u,w])$ is a smooth curve in $\mathbb{R}^2$ of finite length between the points $\mathcal{E}_V([u]_\sim)$ and $\mathcal{E}_V([w]_\sim)$.

We call the resulting tuple $(\mathcal{E}_V(V), \mathcal{E}_E(E))$ an embedding of G^T onto the plane via the embedding function $\mathcal{E}$.

Definition 46: Let $G^T = \bigsqcup_i [n_i, n_{i+1}]/\sim$ be a topological graph. Let V be the endpoints of the intervals $\{[n_i, n_{i+1}]\}$ under the equivalence $\sim$. Furthermore, let E be the set of distinct intervals. An embedding function of G^T onto the (unit) sphere is a tuple of functions $\widetilde{\mathcal{E}} = (\widetilde{\mathcal{E}}_V, \widetilde{\mathcal{E}}_E)$ where

$$\widetilde{\mathcal{E}}_V : V \longrightarrow S^2 \qquad \text{and} \qquad \widetilde{\mathcal{E}}_E : E \longrightarrow \mathcal{P}\left(S^2\right)$$

where $\mathcal{P}\left(S^2\right)$ is the power set, such that for any $v \in V$, $\widetilde{\mathcal{E}}_V(v)$ is a point in the unital sphere, and if $\{u,w\} \in E$ then $\widetilde{\mathcal{E}}_E(\{u,e\})$ is a smooth curve in S^2 of finite length between the points $\widetilde{\mathcal{E}}_V(u)$ and $\widetilde{\mathcal{E}}_V(w)$.

We call the resulting tuple $(\widetilde{\mathcal{E}}_V(V), \widetilde{\mathcal{E}}_E(E))$ an embedding of G^T onto the (unit) sphere via the embedding function $\widetilde{\mathcal{E}}$.

Remark 47: By the fact that any abstract graph can be represented by a topological graph, we can also talk about the embeddings of G (the abstract graph) into the plane or the sphere.

Definition 48: If we do not wish to specify if $\mathcal{E}$ is an embedding function to the plane or to the sphere we will simply call it an embedding function.

Definition 49: Given an embedding function, $\mathcal{E}$, of the abstract graph G we call the set of pre-compositions by isomorphisms of G an non-labeled embedding function. Usually we denote this embedding by $\overline{\mathcal{E}}$. i.e. we have the following:

$$\overline{\mathcal{E}} = (\overline{\mathcal{E}}_E, \overline{\mathcal{E}}_E)$$

where

$$\overline{\mathcal{E}}_E = \{\mathcal{E}_E \circ \lambda | \lambda : G \to G \text{ is a graph isomorphism}\}$$

and

$$\overline{\mathcal{E}}_V = \{\mathcal{E}_V \circ \lambda | \lambda : G \to G \text{ is a graph isomorphism}\}$$

Remark 50: Note that it can be shown that pre-composition creates an equivalence class of embeddings through making the vertices non-labeled in the embedding. Because now, any two labeling on the embedded vertices are equivalent through the relabeling isomorphism on G its self.

Example 51: Now that we have the many interpretations of a graph. It is good to stop and take a look at an example. The one I have chosen is the famous Petersen Graph.

The Petersen Graph can be described as an abstract graph via $P = (V, D, \texttt{rev}, \texttt{head})$ where

- $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$,
- $D = \{(0,1), (0,4), (0,5), (1,0), (1,2), (1,6), (2,1), (2,3), (2,7), (3,2), (3,4), (3,8), (4,0), (4,3), (4,9), (5,0), (5,7), (5,8), (6,1), (6,8), (6,9), (7,2), (7,5), (7,9), (8,3), (8,6), (8,9), (9,4), (9,6), (9,7)\}$,
- $\texttt{rev} : D \to D$ such that $\texttt{rev}(a,b) = (b,a)$ for any $(a,b) \in D$, and
- $\texttt{head} : D \to V$ such that $\texttt{head}(a,b) = b$ for any $(a,b) \in D$.

Similarly, the Petersen Graph can be represented via just vertices and edges, $P = (V, E)$ where

- $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$,
- $E = \{\{0,1\}, \{1,2\}, \{2,3\}, \{3,4\}, \{4,0\}, \{0,5\}, \{1,6\}, \{2,7\}, \{3,8\}, \{4,9\}, \{5,7\}, \{6,8\}, \{7,9\}, \{8,5\}, \{9,6\}\}$

Furthermore, the Petersen Graph can be described as a topological graph via

$$P^T = [0,1] \sqcup [2,3] \sqcup [4,5] \sqcup \cdots \sqcup [28,29] / \sim$$

where

$$\begin{aligned} 0 \equiv 9 \equiv 10, &\quad 11 \equiv 20 \equiv 27, \\ 1 \equiv 2 \equiv 12, &\quad 13 \equiv 22 \equiv 29, \\ 3 \equiv 4 \equiv 14, &\quad 15 \equiv 21 \equiv 24, \\ 5 \equiv 6 \equiv 16, &\quad 17 \equiv 23 \equiv 26, \\ 7 \equiv 8 \equiv 18, &\quad 19 \equiv 25 \equiv 28, \end{aligned}$$

under $\sim$.

And lastly we can consider the following non-labeled embeddings of the Petersen Graph onto the plane:

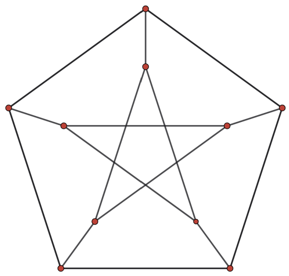


Figure 2: $\overline{\mathcal{E}_1}$

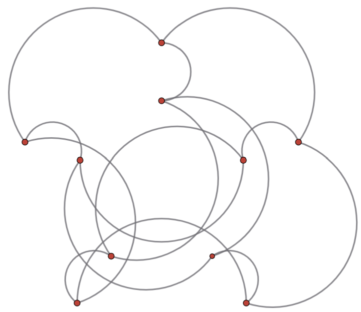


Figure 3: $\overline{\mathcal{E}_2}$

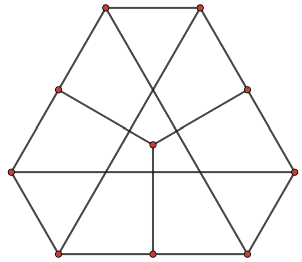


Figure 4: $\overline{\mathcal{E}_3}$

Note that $\overline{\mathcal{E}_{V,1}} = \overline{\mathcal{E}_{V,2}}$ but $\overline{\mathcal{E}_{E,1}} \neq \overline{\mathcal{E}_{E,2}}$. However, $\overline{\mathcal{E}_1}$ and $\overline{\mathcal{E}_2}$ are still distinct non-labeled embeddings. More importantly, even though the embedding of P^T onto the plane via $\overline{\mathcal{E}_1}$ and $\overline{\mathcal{E}_3}$ look completely different, they are both still representing the same graph. However, the embedding via $\overline{\mathcal{E}_1}$ is probably the most famous one.

And lastly, we have can have the following embeddings of P onto the sphere:

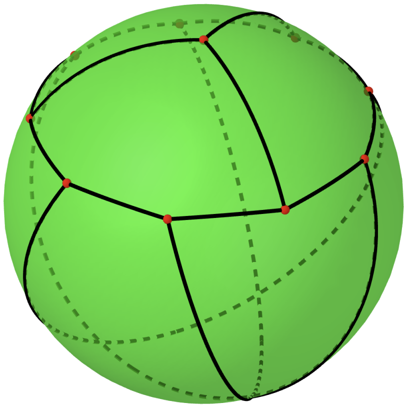


Figure 5: $\overline{\mathcal{E}_4}$

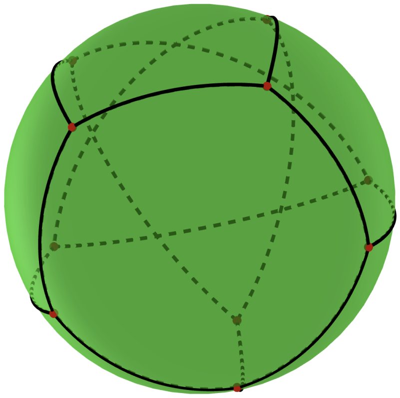


Figure 6: $\overline{\mathcal{E}_5}$

5 Planar Graph Embeddings

Finally, we are able to talk about the theory which is a result of the past three sections of this document. Planar Graphs and their embeddings. How can we find all possible embeddings of a planar graph?

Definition 52: We say that a graph is planar on the plane if it has an embedding onto the plane such that no two edges have non-trivial intersection.

Definition 53: We say that a graph is planar on the sphere if it has an embedding onto the plane such that no two edges have non-trivial intersection.

Remark 54: In this case "non-trivial" mean not at a vertex point.

The reason that we have the two different ways of approaching graph embeddings, is because they are almost the same but not exactly. Even though embeddings on the plane are the most common for graph theory perspectives, it will be relavent in later section to consider the embeddings on the sphere as it tends to be easier to work with.

Theorem 55: A graph is planar on the plane if and only if it is planar on the sphere.

Proof: Let G be a graph which is planar on the plane and let $\mathcal{E}$ be a planar embedding of G onto the plane. Now consider the inverse stereographic projection function on $\mathcal{E}(G)$. Then we get an embedding of G onto the sphere. Note that if this embedding onto the sphere is not planar, then there must exists a point on the sphere which is not a vertex and it lies on two distinct edges. However, then the stereographic projection of that point onto the plane would still be associated with the two edges. Hence meaning that we must have started with a non planar embedding of G onto the plain, which is a contradiction. Thus we know that the inverse stereographic projection of $\mathcal{E}(G)$ results in a planar embedding of G onto the sphere. Note that when we apply the inverse sterographic projection we are missing the point $(0,0,1)$ however, that can simply be added into the sphere since no edge or vertex will pass through the point at infinity of the plane.

Conversely, now suppose that $\overline{\mathcal{E}}$ is a planar embedding of G onto the sphere. By the same reasoning as above, if we take the stereographic projection of G onto the plane via a point not on an edge or vertex, we get a planar embedding of G onto the plane. ∎

Remark 56: Note that this connection is the reason that we call the outer region of a planar embedding of a graph onto the plane a face of the graph. Since in the equivalent planar embedding of the graph on the sphere, that region is an actual bounded region.

Definition 57: We call a graph G planar if it has a planar embedding onto the sphere (hence equivalently, onto the plane).

The reason it is preferred to work in the embedding in the sphere is because then embeddings that look distinct on the planar version are actually the same on the spherical version. But first, let us define what we mean by the "same" embedding. In order to do so, recall the following definition from introductory abstract algebra.

Definition 58: Let X be a set. We define the permutation group of X to be

$$S_X := \{\sigma : X \to X | \sigma \text{ is a bijection}\}$$

with composition as the group operation. We call elements of this set permutations.

Furthermore, that for discrete sets X we can denote a permutation in what is referred to as *cycle notation*. As an example we can consider $X = \{1,2,3,4,5\}$ and the permutation σ that has the mapping $1 \mapsto 3$, $2 \mapsto 5$, $3 \mapsto 1$, $4 \mapsto 2$, $5 \mapsto 4$. In this case we can write $\sigma = (1\ 3)(2\ 5\ 4)$. It can be very easily proven that every permutation can be written as a cycle and every cycle denotes a permutation.

Definition 59: Let $G = (V, D, \texttt{rev}, \texttt{head})$ be a planar graph and $\mathcal{E}$ be a planar embedding of G. The rotation system of $\mathcal{E}$ is the permutation $\sigma \in S_D$ such that for any $d \in D$, if $\texttt{tail}(d) = v$, then $\texttt{tail}(\sigma(d)) = v$ and $\mathcal{E}(\sigma(d))$ is the succeeding dart of $\mathcal{E}(d)$ when following (for this document's purposes) a clockwise order of succession centered around $\mathcal{E}(v)$. If your topological space is the sphere, then view the sphere from the outside.

Here lies the key to understanding "sameness" of planar embeddings. Because we would like to say that two planar embeddings on a particular topological space are equivalent if there exists a homotopy (a map that drags the topological space and none of the edges or vertices get dragged over each other) between the two embeddings. However that is very difficult to keep track of, since there are an uncountable number of homotopies going out of every single embedding of a graph (for which there are uncountably many). Furthermore, it would be difficult to keep track of which embeddings would be planar and which would not be.

Notice that our notion of "sameness" of planar embeddings is mathematically the idea of embeddings being homotopically equivalent to each other, meaning that they have a homotopy that goes from one to the other. However, note that if two planar embeddings are not homotopically equivalent to each other then that means that an edge had to be dragged across another in order to have a "dragging function" between the two. In particular that means that our edge must have been dragged over an edge for which it is incident to. Hence the order for which they appear on the embedding with respect to their shared vertex must have changed. In other words, if two embedding are not homotopically equivalent, then their rotation systems would be distinct.

Hence, we can build an equivalence class of embeddings via the rotation system of embeddings. However, there is one caveat to this. Rotation system equivalences of embeddings onto the *plane* include embeddings which are not homotopically equivalent. This is because Rotation systems for planar graphs assume that you can move your edge in any direction so long as you do not cross other edges. However, on the plane you are not able to bend an edge all the way to infinity then bring it back on the other side. Hence you are not always able to move edges in the way Rotation systems wish you can.

Note that this issue is not with the equivalence classes. Because technically speaking, allowing you to bend an edge through infinity would still preserve the faces of an embedding (more detail later). Hence it is more convenient to consider the equivalent embedding onto the sphere, then project it down into the plane. Because on the sphere, there is no issues with having to bend an edge up to infinity, because there is no infinity on the topology of the sphere.

As a last remark, we should mention the fact that the way we project a sphere onto the plane is via the stereographic projection. However when projecting, we are (in reality) poling a hole in the sphere and opening it to view the insides of the sphere. Note that this is the opposite of the definition of a rotation system of an embedding onto a sphere, hence all of our rotations are now reversed. Thus, in order to maintain the rotation system between a sphere and its planar representation, we need to perform a reflection (across an arbitrary line) to reverse the rotation back into a clockwise motion.

Now, let us look at how to actually find all possible embeddings of a graph.

Definition 60: Let G be a graph with planar embedding $\mathcal{E}$ onto $\mathcal{T}$ (either sphere or plane). A face of G under the embedding $\mathcal{E}$ is an open set in $\mathcal{T} \setminus \mathtt{Im}(\mathcal{E})$. We denote the set of faces by F.

Remark 61: Note that any planar embedding can be identified by three factors, the vertices, the edges, and the faces. Hence, if we can find a way to represent a planar graph alongside its rotation system, we are able to account for all planar embeddings up to homotopy.

Definition 62: Let $G = (V, D, \mathtt{rev}, \mathtt{head})$ be a graph and let $\mathcal{E}$ be a planar embedding of G. Furthermore, let $d \in D$ be an arbitrary dart of G. If $f \in F$ is a face such that $\mathcal{E}(d) \subseteq \partial f$ then, f is called a shore of d.

Remark 63: Note that by the way we have defined embeddings of darts (and edges), darts are curves with direction in our directed spaces. Hence it follows that every dart has exactly two shores.

Definition 64: Let $G = (V, D, \mathtt{rev}, \mathtt{head})$ be a graph and let $\mathcal{E}$ be a planar embedding of G. We define the following functions:

- Let $\mathtt{left} : D \to F$ such that $\mathtt{left}(d) = f$ where f is to the left of d when following the direction of $\mathcal{E}(d)$.
- Let $\mathtt{right} : D \to F$ such that $\mathtt{right}(d) = f$ where f is to the right of d when following the direction of $\mathcal{E}(d)$.

Definition 65: Let $G = (V, D, \mathtt{rev}, \mathtt{head})$ be a graph and let $\mathcal{E}$ be a planar embedding of G. For $f \in F$ we define the degree of f to be

$$\mathtt{deg}(f) := |\{d \in D | \mathtt{right}(d) = f\}|$$

The next steps are done so that we can precisely talk about both planar graphs and embeddings simultaneously.

Definition 66: A combinatorial map of a graph G is a triple $\Sigma = (D, \alpha, \sigma)$ where D is the set of darts, α is an involution of D with no fixed points, and σ is a permutation of D. Note that we can consider σ, α, and $\sigma \circ \alpha$ to be acting on D via the groups that they generate. For a particular combinatorial map Σ we define

- The vertices of Σ are the orbits of σ,
- The edges of Σ are the orbits of α,
- the faces of Σ are the orbits of $\alpha \circ \sigma$.

Now, note that if we choose our combinatorial map for a graph G to be $\Sigma = (D, \sigma, \texttt{rev})$ where σ is an element of S_D such that $\texttt{tail}(\sigma(d)) = \texttt{tail}(d)$ then we have an abstract representation of our graph which captures the ideas of vertices, edges, and faces in a countable way. As a result, we can use Euler's Characteristic equation

$$|V| - |E| + |F| = 2$$

to determine if the chosen σ results in a planar graph. Note that choosing σ is equivalent to choosing an embedding of G, because it is representing the rotation system of a class of embeddings.

As such we get the following protocol for finding all possible embeddings of a graph:

0. Identify your given graph G as an abstract graph $G = (V, D, \texttt{rev}, \texttt{head})$. This means that if G was non-labled, you will have to label it.
1. Define $\alpha \in S_D$ to be the permutation made by composition of all translations of the form $(d \quad \texttt{rev}(d))$.
2. Find all permutations $\sigma \in S_D$ such that for any $d \in D$, $\texttt{tail}(\sigma(d)) = \texttt{tail}(d)$.
3. Compute $\alpha \circ \sigma$.
4. Compute the orbits of σ, α, and $\alpha \circ \sigma$, and see if they satisfy Euler's characteristic equation.
5. All combinatorial maps that satisfy Euler's characteristic equation are non homotopically equivalent embeddings of G **with labels**.
6. If you would like to identify all non-homotopically equivalent embeddings of G you will have to build an equivalence class under isomorphisms of G.

6 The Markov Chain

We now turn our attention to [MP15]. The paper first introduces the Markov chain $\mathcal{M}_{\text{path}}$ which is used to sample st-paths within a 2-connected planar graph. This is how it works: Let $G = (v, E)$ be a planar graph, and let $\mathcal{E}$ be a planar embedding of G. Let $s, t \in V$ be distinct vertices. The process will sample st-paths in G. During the sampling process we will choose a particular $\lambda \in \mathbb{R}_{>0}$. Now, given a particular path $\mathbf{x} \in \Omega$, to determine which new st-path to move to, follow the following algorithm:

Algorithm 1 Process of changing st-paths using the Markov chain $\mathcal{M}_{\text{path}}$

Require: $\lambda > 0$, $\mathbf{x} \in \Omega$.
With probability $\frac{1}{2}$ do nothing.
if You decided not to do nothing **then**
  Choose a face uniformly at random.
  if The chosen face is adjacent to $\mathbf{x}$ **then**
    Re-rout $\mathbf{x}$ through the chosen face to get the path $\mathbf{y}$ with probability

$$A(\mathbf{x}, \mathbf{y}) = \min\left\{1, \frac{\lambda^{|y|}}{\lambda^{|x|}}\right\}$$

  else
    Do nothing.
  end if
else
  Do nothing.
end if

Now in order for us to find the actual probability matrix of $\mathcal{M}_{\text{path}}$ let us have the following notation:

1. Let Ω be the set of all st-paths, this will be our sample space.
2. Let F be all the faces of the planar embedding $\mathcal{E}$. Note that if we are embedding onto the plane then this includes the unbounded face. This is not a requirement, even the authors of [MP15] exclude it. However, it does not change the math too much to keep it.
3. Let n be our discrete time variable.
4. Let $\mathcal{N}_n$ be random variables with state space $\{0,1\}$ such that $\mathcal{N}_n \sim$ Uniform for all n.
5. Let $\mathcal{F}_n$ be random variables with state space F such that $\mathcal{F}_n \sim$ Uniform for all n.
6. Let $\mathcal{Y}_n$ be random variables which determine the chosen st-path at time n. Note that the state space of $\mathcal{Y}_n$ is Ω.

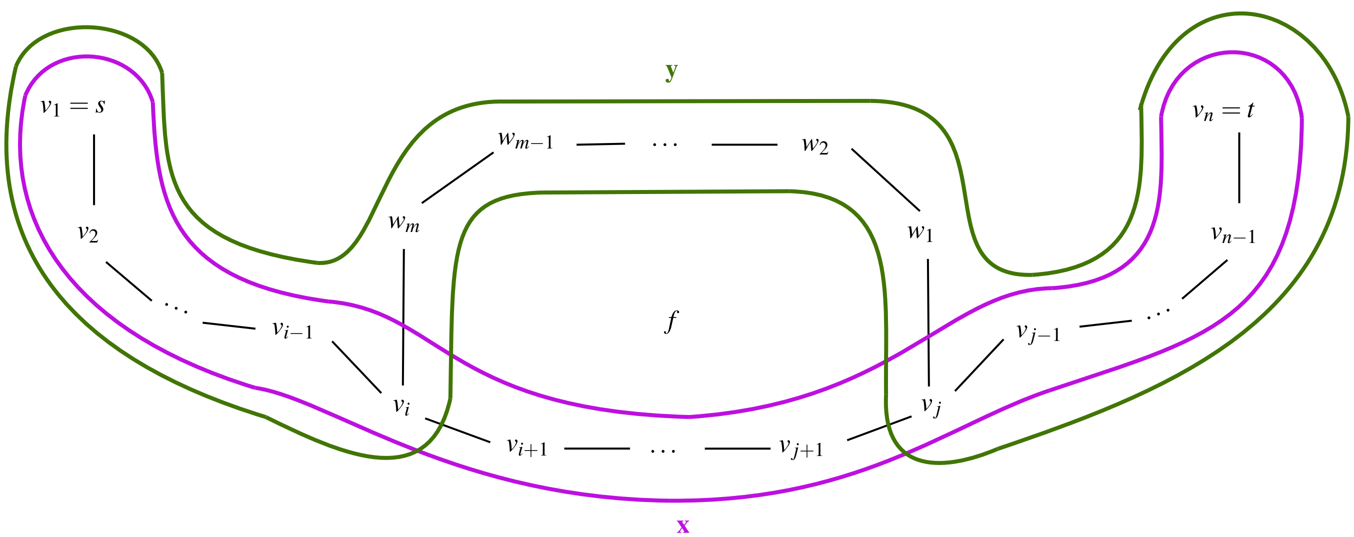
[MP15] uses this idea of re-routing an st-path through a face. To make this more precise we use the following definitions.

Definition 67: Let $\mathbf{x} \in \Omega$ and let $f \in F$. We say that $\mathbf{x}$ is adjacent to f (or f is adjacent to $\mathbf{x}$) if the intersection of the edges of $\mathbf{x}$ and the boundary edges of f makes a single path of length at least 1.

Remark 68: Note that this definition excludes the case when a path $\mathbf{x}$ intersects the boundary edges of f via two disconnected sub-paths.

Definition 69: Suppose $\mathbf{x} = \{v_1 = s, v_2, \ldots, v_n = t\} \in \Omega$ and $f \in F$ such that $\mathbf{x}$ is adjacent to f. Hence we know that there exists $1 \leq i < j \leq n$ such that $P = \{v_i, v_{i+1}, \ldots, v_j\}$ is the unique path graph made by the intersection of the edges of f and the edges of $\mathbf{x}$. This means that the cycle graph that makes up the boundary edges of f is of the form $C = \{v_i, v_{i+1}, \ldots, v_j, w_1, w_2, \ldots, w_m, v_i\}$. We define the re-routing of $\mathbf{x}$ through the face f as the new path

$$\mathbf{y} = \{v_1 = s, v_2, \ldots, v_i, w_m, w_{m-1}, \ldots, w_1, v_j, v_{j+1}, \ldots, v_n = t\}$$



Hence let us be more concrete with the way that we approach re-routing of paths. Let the following notation hold:

1. Define the function $I : \Omega \times F \to \Omega$ such that

$$(\mathbf{x}, f) \mapsto \begin{cases} y & \text{if } \mathbf{x} \text{ is adjacent to } f \\ \emptyset & \text{otherwise} \end{cases}$$

where $\mathbf{y}$ is the path made by re-routing $\mathbf{x}$ through the face f.

2. For each $\mathbf{x} \in \Omega$ define the function $I_{\mathbf{x}} : \Omega \to \Omega$ such that $f \mapsto I(\mathbf{x}, f)$.
3. Let $\mathbf{x}, \mathbf{y} \in \Omega$. We say that $\mathbf{x}$ is adjacent to $\mathbf{y}$ if there exists a face $f \in F$ which is adjacent to $\mathbf{x}$ such that $I(\mathbf{x}, f) = \mathbf{y}$.
4. $\mathcal{L}_{\mathbf{x}} := \{\mathbf{y} \in \mathtt{Im}(I_{\mathbf{x}}) : |\mathbf{y}| > |\mathbf{x}|\}$
5. $\overline{\mathcal{L}}_{\mathbf{x}} := \{\mathbf{y} \in \mathtt{Im}(I_{\mathbf{x}}) : |\mathbf{y}| \geq |\mathbf{x}|\}$
6. $\mathcal{S}_{\mathbf{x}} := \{\mathbf{y} \in \mathtt{Im}(I_{\mathbf{x}}) : |\mathbf{y}| < |\mathbf{x}|\}$
7. $\overline{\mathcal{S}}_{\mathbf{x}} := \{\mathbf{y} \in \mathtt{Im}(I_{\mathbf{x}}) : |\mathbf{y}| \leq |\mathbf{x}|\}$

Now suppose that we are given the following:

1. a particular embedding, $\mathcal{E}$, of the planar graph G,
2. $\mathbf{x}, \mathbf{y} \in \Omega$,
3. $Y_n = \mathbf{x}$

Now, note that there are two cases for how $\mathbf{x}$ and $\mathbf{y}$ relate to each other:

Case 1: $\mathbf{y} \neq \mathbf{x}$, in which case we could have

Case 1.1: $\mathbf{y} \in \mathtt{Im}(I_{\mathbf{x}})$, or

Case 1.2: $\mathbf{y} \notin \mathtt{Im}(I_{\mathbf{x}})$.

Case 2: $\mathbf{y} = \mathbf{x}$.

Let us look at Case 2 first:

Case 2: Note that we have the following:

$$P(Y_{n+1} = \mathbf{y} | Y_n = \mathbf{x}) = P(Y_{n+1} = \mathbf{x} | Y_n = \mathbf{x}) = 1 - \sum_{\mathbf{z} \in \Omega \setminus \{\mathbf{x}\}} P(Y_{n+1} = \mathbf{z} | Y_n = \mathbf{x}) \tag{1}$$

Hence we know that we can find the probability in this case by understanding what happens in Case 1.

Case 1.2: Suppose $\mathbf{y} \notin \mathtt{Im}(I_{\mathbf{x}})$. By definition this implies that there does not exist a face $f \in F$ such that $I(\mathbf{x}, f) = \mathbf{y}$. Hence we know by Step 7 of Algorithm 1 that when $Y_n = \mathbf{x}$ then $Y_{n+1} = \mathbf{x}$. As such, we know that

$$P(Y_{n+1} = \mathbf{y} | Y_n = \mathbf{x}) = 0 \tag{2}$$

Case 1.1: Suppose $\mathbf{y} \notin \mathtt{Im}(I_{\mathbf{x}})$ and $\mathbf{y} \neq \mathbf{x}$. By our first supposition we know that there exists $f \in F$ such that $\mathbf{y} = I(\mathbf{x}, f)$, let f be precisely that face. Note that Algorithm 1 tells us the following pieces of information:

$$P(Y_{n+1} = \mathbf{y}, N_{n+1} = 0 | Y_n = \mathbf{x}) = 0 \tag{3}$$

$$P(Y_{n+1} = \mathbf{y}, F_{n+1} \neq f | Y_n = \mathbf{x}, N_{t+1} = 1) = 0 \tag{4}$$

$$P(N_n = 0) = P(N_n = 1) = \frac{1}{2} \qquad \text{for any } n \in \mathbb{N} \tag{5}$$

$$P(F_n = g | N_n = 1) = \frac{1}{|F|} \qquad \text{for any } g \in F \text{ and for any } n \in \mathbb{N} \tag{6}$$

$$P(Y_{n+1} = \mathbf{y} | Y_n = \mathbf{x}, N_{n+1} = 1, F_{n+1} = f) = A(\mathbf{x}, \mathbf{y}) \tag{7}$$

$$N_n \perp Y_n \tag{8}$$
$$F_n \perp Y_n \tag{9}$$

Hence we have the following string of equalities in this case:

$$\begin{aligned}
P(Y_{n+1}=\mathbf{y}|Y_n=\mathbf{x}) &= P(Y_{n+1}=\mathbf{y},N_{n+1}=0|Y_n=\mathbf{x})+P(Y_{n+1}=\mathbf{y},N_{n+1}=1|Y_n=\mathbf{x})\\
&= 0+P(Y_{n+1}=\mathbf{y},N_{n+1}=1|Y_n=\mathbf{x}) && \text{(by (3))}\\
&= P(Y_{n+1}=\mathbf{y},N_{n+1}=1|Y_n=\mathbf{x})\\
&= P(Y_{n+1}=\mathbf{y}|Y_n=\mathbf{x},N_{n+1}=1)P(N_{n+1}=1|Y_n=\mathbf{x})\\
&= P(Y_{n+1}=\mathbf{y}|Y_n=\mathbf{x},N_{n+1}=1)P(N_{n+1}=1) && \text{(by (8))}\\
&= \frac{1}{2}P(Y_{n+1}=\mathbf{y}|Y_n=\mathbf{x},N_{n+1}=1) && \text{(by (5))}\\
&= \frac{1}{2}(P(Y_{n+1}=\mathbf{y},F_{n+1}\neq f|Y_n=\mathbf{x},N_{n+1}=1)+P(Y_{n+1}=\mathbf{y},F_{n+1}=f|Y_n=\mathbf{x},N_{n+1}=1))\\
&= \frac{1}{2}(0+P(Y_{n+1}=\mathbf{y},F_{n+1}=f|Y_n=\mathbf{x},N_{n+1}=1)) && \text{(by (4))}\\
&= \frac{1}{2}P(Y_{n+1}=\mathbf{y},F_{n+1}=f|Y_n=\mathbf{x},N_{n+1}=1)\\
&= \frac{1}{2}P(Y_{n+1}=\mathbf{y}|Y_n=\mathbf{x},N_{n+1}=1,F_{n+1}=f)P(F_{n+1}=f|Y_n=\mathbf{x},N_{n+1}=1)\\
&= \frac{1}{2}P(Y_{n+1}=\mathbf{y}|Y_n=\mathbf{x},N_{n+1}=1,F_{n+1}=f)P(F_{n+1}=f|N_{n+1}=1) && \text{(by (9))}\\
&= \frac{1}{2|F|}P(Y_{n+1}=\mathbf{y}|Y_n=\mathbf{x},N_{n+1}=1,F_{n+1}=f) && \text{(by (6))}\\
&= \frac{1}{2|F|}A(\mathbf{x},\mathbf{y}) && \text{(by (7))}\\
&= \frac{1}{2|F|}\min\left\{1,\lambda^{|\mathbf{y}|-|\mathbf{x}|}\right\}
\end{aligned}$$

Hence we know that if $\mathbf{y}\in\mathtt{Im}(I_{\mathbf{x}})$ and $\mathbf{y}\neq\mathbf{x}$, then

$$P(Y_{n+1}=\mathbf{y}|Y_n=\mathbf{x})=\frac{1}{2|F|}\min\left\{1,\lambda^{|\mathbf{y}|-|\mathbf{x}|}\right\} \tag{10}$$

Thus we can use the information we have to evaluate <u>case 2</u>:

$$\begin{aligned}
P(Y_{n+1}=\mathbf{x}|Y_n=\mathbf{x}) &= 1-\sum_{\mathbf{z}\in\Omega\setminus\{\mathbf{x}\}}P(Y_{n+1}=\mathbf{z}|Y_n=\mathbf{x}) && \text{(by (1))}\\
&= 1-\left(\sum_{\mathbf{z}\notin\mathtt{Im}(I_{\mathbf{x}})}P(Y_{n+1}=\mathbf{z}|Y_n=\mathbf{x})+\sum_{\mathbf{z}\in\mathtt{Im}(I_{\mathbf{x}})\setminus\{\mathbf{x}\}}P(Y_{n+1}=\mathbf{z}|Y_n=\mathbf{x})\right)\\
&= 1-\sum_{\mathbf{z}\notin\mathtt{Im}(I_{\mathbf{x}})}P(Y_{n+1}=\mathbf{z}|Y_n=\mathbf{x})-\sum_{\mathbf{z}\in\mathtt{Im}(I_{\mathbf{x}})\setminus\{\mathbf{x}\}}P(Y_{n+1}=\mathbf{z}|Y_n=\mathbf{x})\\
&= 1-0-\sum_{\mathbf{z}\in\mathtt{Im}(I_{\mathbf{x}})\setminus\{\mathbf{x}\}}P(Y_{n+1}=\mathbf{z}|Y_n=\mathbf{x}) && \text{(by (2))}\\
&= 1-\sum_{\mathbf{z}\in\mathtt{Im}(I_{\mathbf{x}})\setminus\{\mathbf{x}\}}\frac{1}{2|F|}\min\left\{1,\lambda^{|\mathbf{y}|-|\mathbf{x}|}\right\} && \text{(by (10))}\\
&= 1-\frac{1}{2|F|}\sum_{\mathbf{z}\in\mathtt{Im}(I_{\mathbf{x}})\setminus\{\mathbf{x}\}}\min\left\{1,\lambda^{|\mathbf{y}|-|\mathbf{x}|}\right\}
\end{aligned}$$

Now let us try to understand the minimum function a bit better. Note that the minimum function depends on three variables, $\lambda,|\mathbf{y}|$, and $|\mathbf{x}|$. This turns out to result in five different cases:

Case i: If $\lambda = 1$, then $\min\{1, \lambda^{|\mathbf{y}|-|\mathbf{x}|}\} = 1$.

Case ii: If $|\mathbf{y}| \geq |\mathbf{x}|$ and $\lambda > 1$, then $\min\{1, \lambda^{|\mathbf{y}|-|\mathbf{x}|}\} = 1$.

Case iii: If $|\mathbf{y}| < |\mathbf{x}|$ and $\lambda > 1$, then $\min\{1, \lambda^{|\mathbf{y}|-|\mathbf{x}|}\} = \lambda^{|\mathbf{y}|-|\mathbf{x}|} < 1$

Case iv: If $|\mathbf{y}| \leq |\mathbf{x}|$ and $\lambda < 1$, then $\min\{1, \lambda^{|\mathbf{y}|-|\mathbf{x}|}\} = 1$.

Case v: If $|\mathbf{y}| > |\mathbf{x}|$ and $\lambda > 1$, then $\min\{1, \lambda^{|\mathbf{y}|-|\mathbf{x}|}\} = \lambda^{|\mathbf{y}|-|\mathbf{x}|} < 1$

Now with this information we can be as explicit as possible. Note that we have the following computations:

If $\lambda > 1$:

$$P(Y_{n+1} = \mathbf{y} | Y_n = \mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{y} \notin \mathtt{Im}(I_{\mathbf{x}}) \\ \dfrac{1}{2|F|} & \text{if } \mathbf{y} \in \overline{\mathcal{L}}_{\mathbf{x}} \\ \dfrac{\lambda^{|\mathbf{y}|-|\mathbf{x}|}}{2|F|} & \text{if } \mathbf{y} \in \mathcal{S}_{\mathbf{x}} \\ 1 - \dfrac{1}{2|F|}\left(|\overline{\mathcal{L}}_{\mathbf{x}}| - \displaystyle\sum_{\mathbf{z} \in \mathcal{S}_{\mathbf{x}}} \lambda^{|\mathbf{z}|-|\mathbf{x}|}\right) & \text{if } \mathbf{y} = \mathbf{x} \end{cases}$$

If $\lambda < 1$:

$$P(Y_{n+1} = \mathbf{y} | Y_n = \mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{y} \notin \mathtt{Im}(I_{\mathbf{x}}) \\ \dfrac{1}{2|F|} & \text{if } \mathbf{y} \in \overline{\mathcal{S}}_{\mathbf{x}} \\ \dfrac{\lambda^{|\mathbf{y}|-|\mathbf{x}|}}{2|F|} & \text{if } \mathbf{y} \in \mathcal{L}_{\mathbf{x}} \\ 1 - \dfrac{1}{2|F|}\left(|\overline{\mathcal{S}}_{\mathbf{x}}| - \displaystyle\sum_{\mathbf{z} \in \mathcal{L}_{\mathbf{x}}} \lambda^{|\mathbf{z}|-|\mathbf{x}|}\right) & \text{if } \mathbf{y} = \mathbf{x} \end{cases}$$

If $\lambda = 1$:

$$P(Y_{n+1} = \mathbf{y} | Y_n = \mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{y} \notin \mathtt{Im}(I_{\mathbf{x}}) \\ \dfrac{1}{2|F|} & \text{if } \mathbf{y} \in \overline{\mathcal{S}}_{\mathbf{x}} \\ 1 - \dfrac{|\mathtt{Im}(I_{\mathbf{x}})|}{2|F|} & \text{if } \mathbf{y} = \mathbf{x} \end{cases}$$

Note that with this information it becomes fairly simple to write code which takes a graph with an embedding as input and outputs the probability matrix of the Markov chain associated to it. In fact this has already been done by the first author and is publicly available in [Hed25]. Please note that the code is not optimized and is coded on the coding language SageMath. The reason for not optimizing is due to the fact that SageMath is not an optimized language hence the efforts would not be very rewarding.

6.1 Conclusion and Future Work

Note that the only reason most of this document exists is due to two important theorems in [MP15]. First is the bound which they provide for the diameter of the graph of the Markov chain:

Theorem 70: For any plane embedding of a given graph and any pair of vertices s and t, the Markov chain $\mathcal{M}_{\text{path}}$ has diameter at most $2n^2$, where n is the number of vertices in the graph which is getting sampled.

Second is the mixing time of a slightly altered Markov chain which they denoted $\mathcal{M}^*_{\text{mon}}$.

Theorem 71: The mixing time of $\mathcal{M}^*_{\text{mon}}$ is $O(n^3/\lambda^2)$ for every $\lambda \in (0,1]$.

While their result is very useful, Theorem 71's proof depended on the bound of the diameter of the Markov chain $\mathcal{M}_{\text{path}}$. This bound is not only very large compared to the actual diameter of most graphs, but the bound also depends on the embedding of the graph we are sampling from.

Hence what may be good for future work is to explore the connection between the different embeddings a palanar graph has and the diameter of the graphical representation of the Markov chain $\mathcal{M}_{\text{path}}$

References

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